Constraint Programming (CP) coupled with Interval Analysis (IA) for mobile robotics

(a very short introduction)

Simon Rohou, Quentin Brateau

ENSTA Bretagne, Lab-STICC, UMR CNRS 6285, Brest, France

Journées Nationales de la Recherche en Robotique (JNRR 2023)
Moliets 16–20th October 2023

Planning de cet après-midi

- **14h00 – 15h00**
  Présentation
- **15h00 – 16h00**
  TD intervals .. static range-only
- **16h00 – 16h30**
  Pause café
- **16h30 – 17h00**
  Présentation
- **17h00 – 18h00**
  TD dynamic localization .. SLAM

Section 1

Introduction

Mobile robotics

Robot localization = state estimation problem.
Classically, we have:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) & \text{(evolution)} \\ \end{cases}$$

Where:

- $\mathbf{x} \in \mathbb{R}^n$ is the state vector (position, bearing, ...)
- $\mathbf{u} \in \mathbb{R}^m$ is the input vector (command)
- $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the *evolution* function

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$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) & \text{(evolution)} \\ \mathbf{z}(t) = \mathbf{g}\big(\mathbf{x}(t)\big) & \text{(observation)} \end{cases}$$

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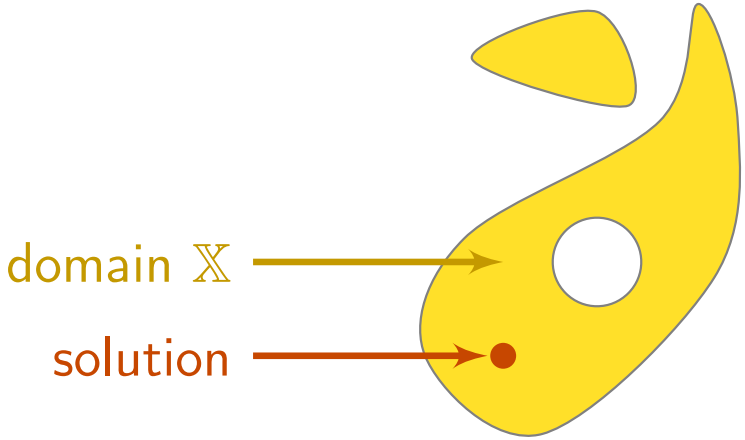
- $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^p$ is the *observation* function
- $\mathbf{z} \in \mathbb{R}^p$ is some exteroceptive measurement (camera, sonar...)

Constraint programming: overall concept

- system described by a *constraint network*
- **variables** belonging to **domains** $\mathbb{X}$



Constraint network:

$$\begin{cases} \textbf{Variables: } \mathbf{x} \\ \textbf{Constraints:} \\ \\ \textbf{Domains: } \mathbb{X} \end{cases}$$

Contractor Programming
Chabert, Jaulin *Artifical Intelligence*, 2009

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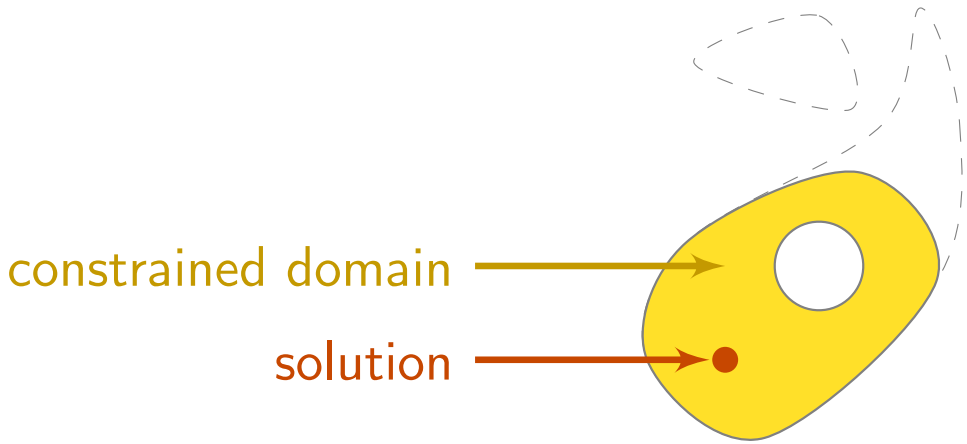

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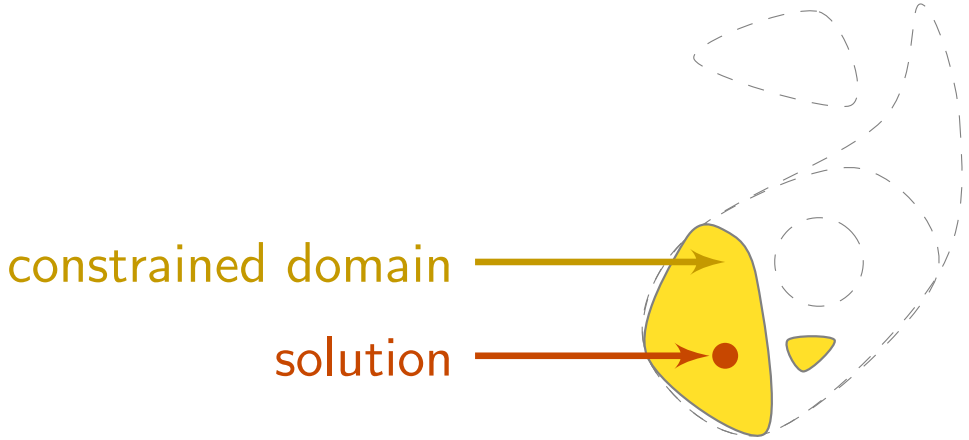

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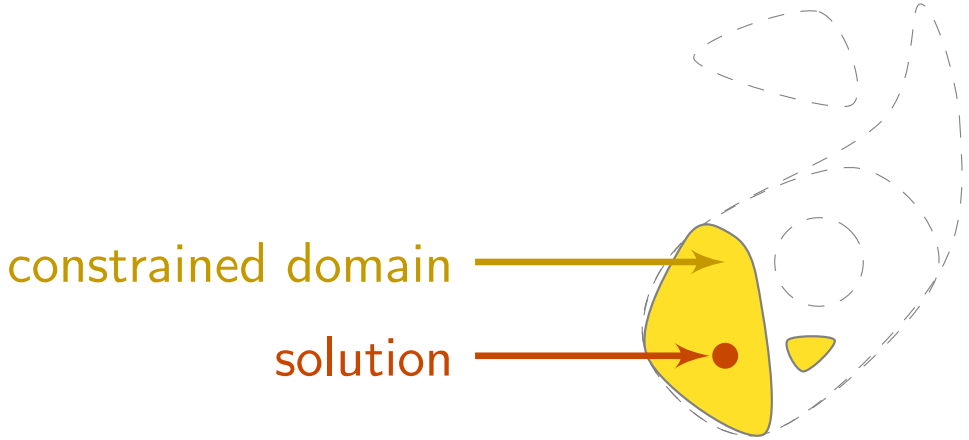

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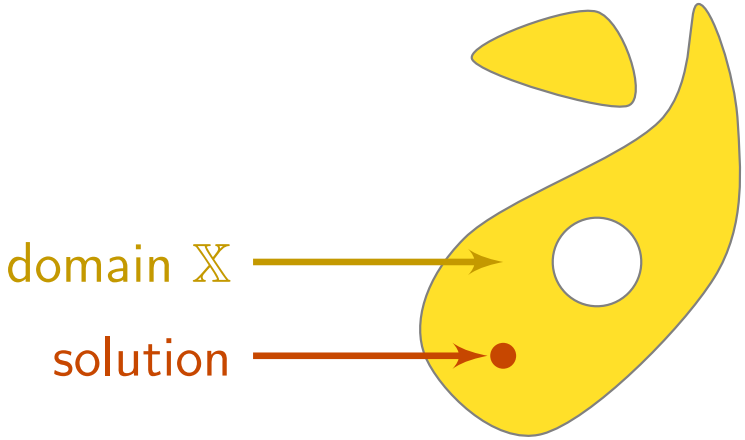

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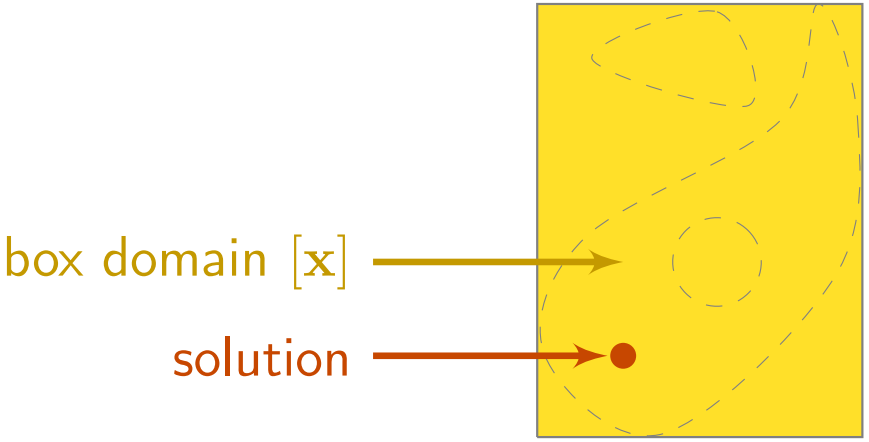

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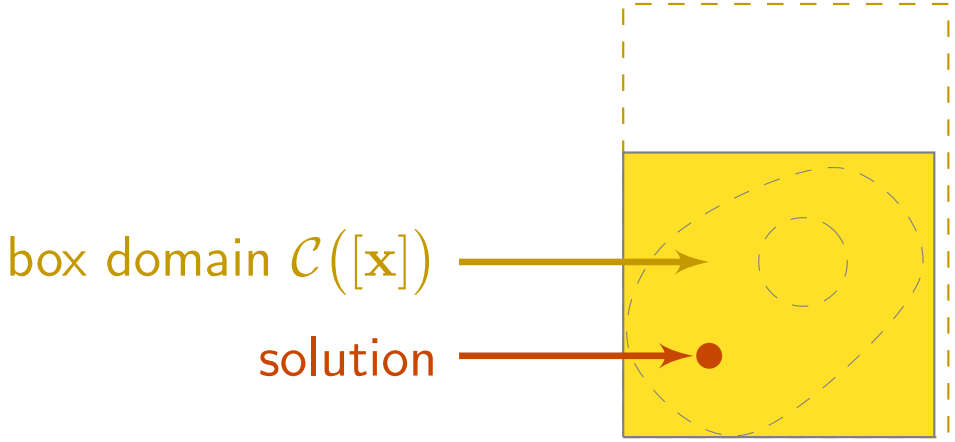

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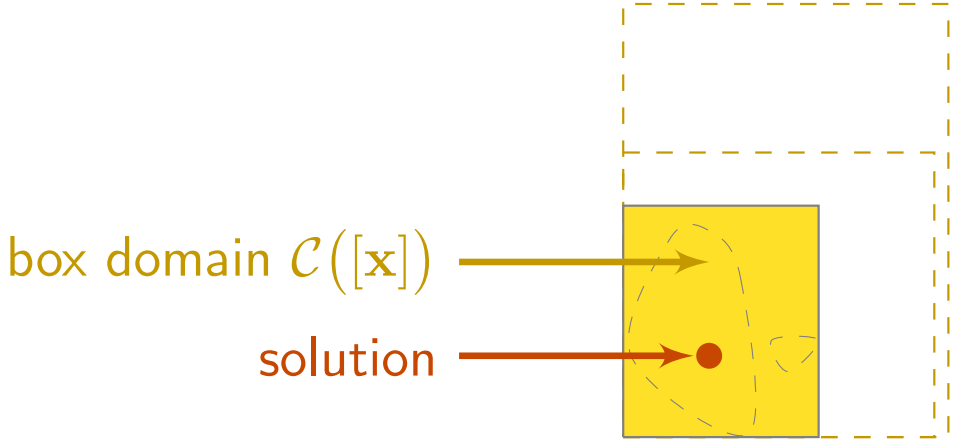

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Section 2

Interval analysis

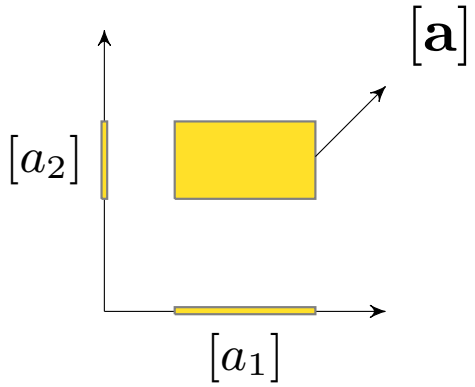
What is an interval?

An interval $[x]$:

- a closed and connected subset of $\mathbb{R}$ delimited by two bounds
- $[x] = [x^-, x^+] = \{x \in \mathbb{R} \mid x^- \leqslant x \leqslant x^+\}$
- $[x] \in \mathbb{IR}$

A box $[\mathbf{x}]$ (an interval vector):

- a cartesian product of n intervals
- $[\mathbf{x}] \in \mathbb{IR}^n$



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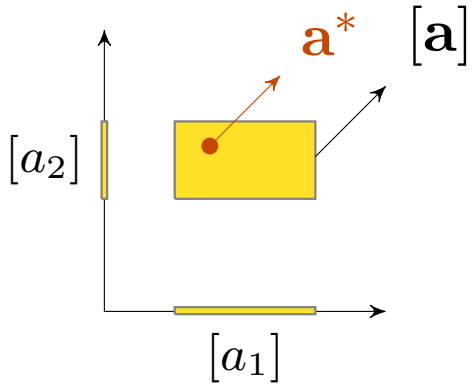

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Interval Analysis

IA is based on the extension of all classical **real arithmetic operators**:

- $+, -, \times, \div$

  ex: $[x] + [y] = [x^- + y^-, x^+ + y^+]$

  ex: $[x] - [y] = [x^- - y^+, x^+ - y^-]$

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Adaptation of **elementary functions** such as:

- $\cos$, $\exp$, $\tan$, etc.
- output is the smallest interval containing all the images of all defined inputs through the function

Example: $\exp([x])$

Example of a **forward evaluation** of $\exp([x])$:

Natural inclusion functions

Example of the previous function g.
Let us compute the **inclusion function** of the distance function g:
(distance between a position $\mathbf{x}$ and a landmark $\mathbf{b}$)

$$\begin{array}{rcl} g : \mathbb{R}^2 & \rightarrow & \mathbb{R}, \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \mapsto & \sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2}. \end{array} \tag{1}$$

Replacing items by their interval counterpart, $[g]$ is given by:

$$\begin{array}{rcl} [g] : \mathbb{IR}^2 & \rightarrow & \mathbb{IR}, \\ \begin{pmatrix} [x_1] \\ [x_2] \end{pmatrix} & \mapsto & \sqrt{([x_1] - [b_1])^2 + ([x_2] - [b_2])^2}. \end{array} \tag{2}$$

Forward/Backward with interval analysis

Information can be propagated in a forward and a backward way in the equation.

Recall the example of $\exp$:

- $y = \exp(x)$
- $x \in [x]$, $y \in [y]$

Section 3

Constraint Propagation (CP) coupled with Interval Analysis(IA)

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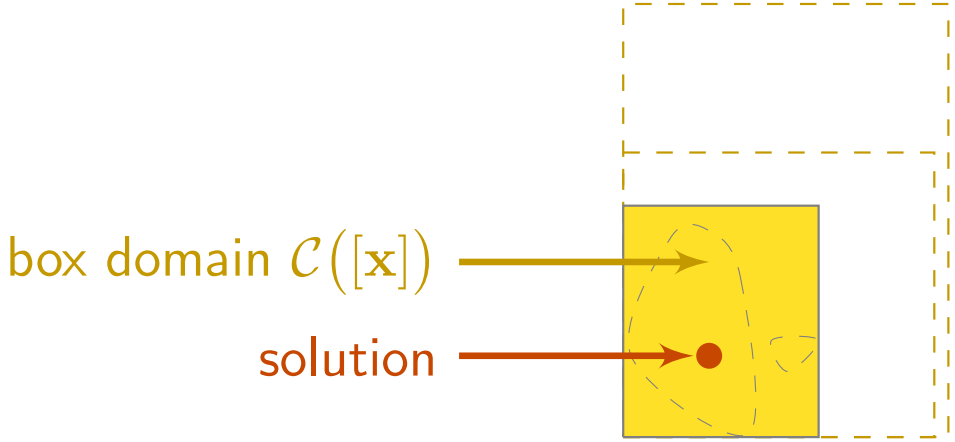

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Contractors

Considering a constraint (*e.g.*, an equation) $\mathcal{L}$,

Definition

A contractor on a box is an operator $\mathcal{C}_{\mathcal{L}}$ from $\mathbb{IR}^n$ to $\mathbb{IR}^n$ such that:

$$(i) \quad \forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathcal{C}_{\mathcal{L}}([\mathbf{x}]) \subseteq [\mathbf{x}], \qquad \text{(contraction)}$$

$$(ii) \quad \begin{pmatrix} \mathcal{L}(\mathbf{x}) \\ \mathbf{x} \in [\mathbf{x}] \end{pmatrix} \Longrightarrow \mathbf{x} \in \mathcal{C}_{\mathcal{L}}([\mathbf{x}]). \qquad \text{(consistency)}$$

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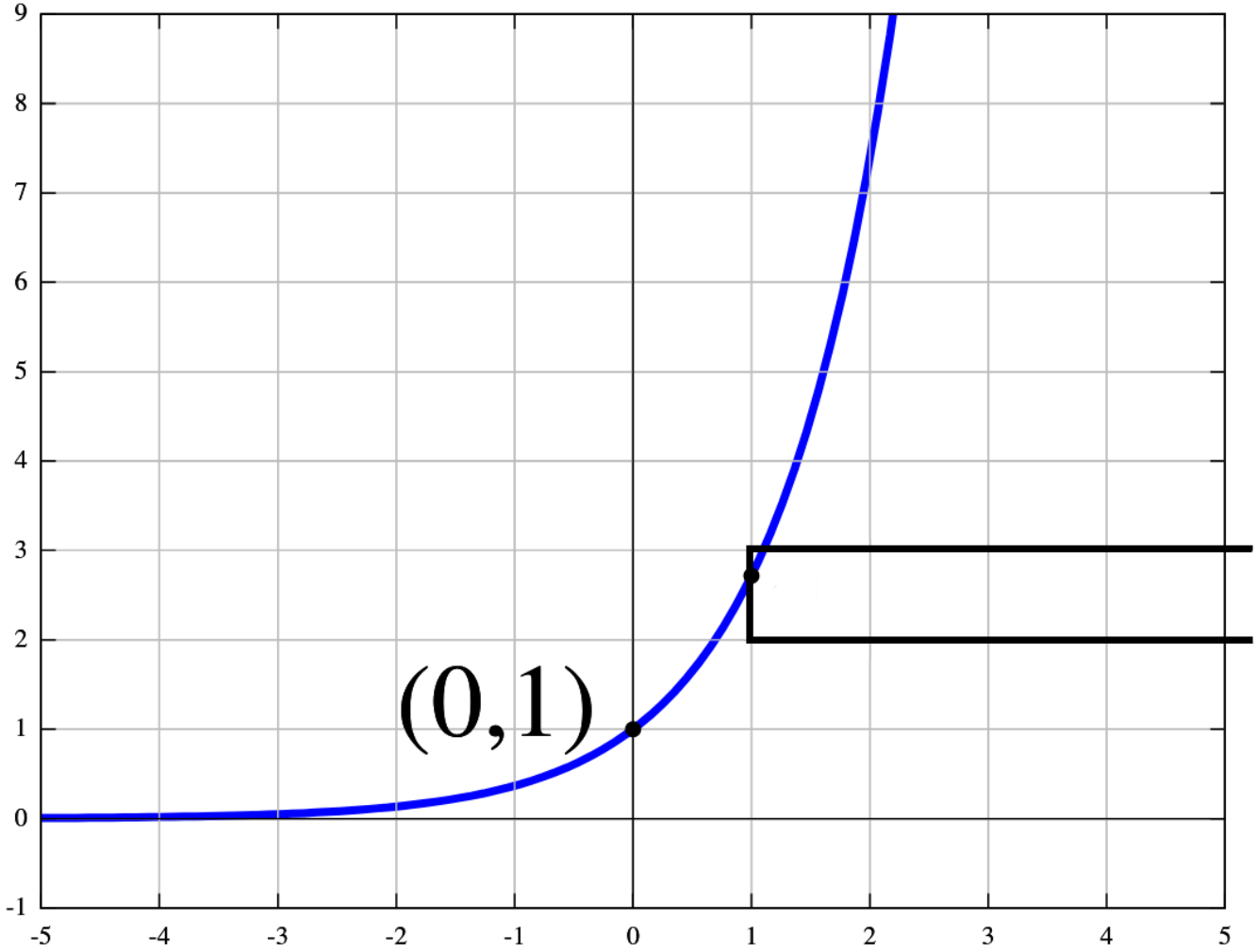
Example:

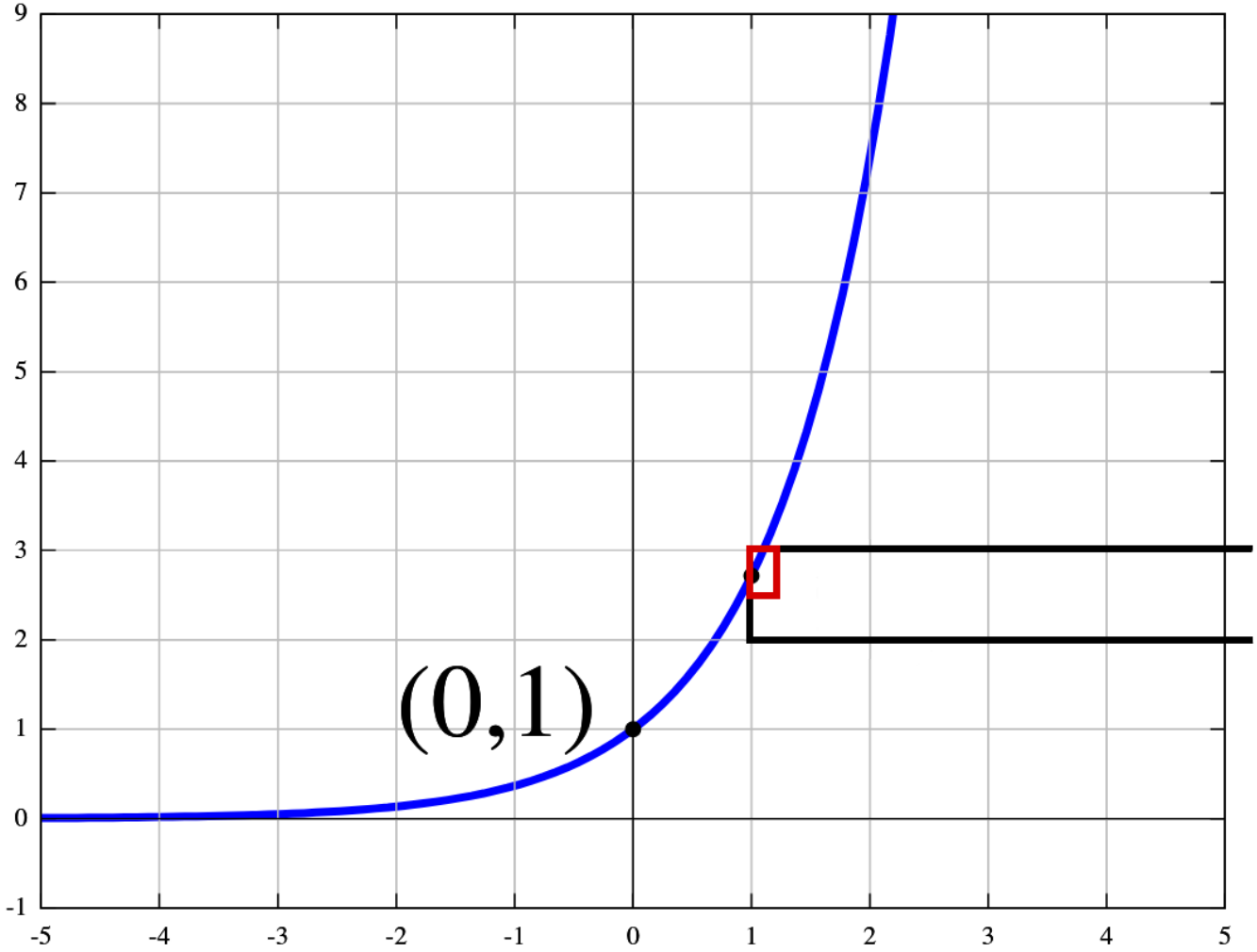
- $[\mathbf{x}] = [1, \infty] \times [2, 3]$
- $\mathcal{C}_{\exp}([\mathbf{x}])$ associated with $\exp(x_1) - x_2 = 0$
- after applying $\mathcal{C}_{\exp}$, $[\mathbf{x}] = [1, 1.099] \times [2.72, 3]$

Contractors



Black: initial box $[1, \infty] \times [2, 3]$.

Contractors



Black: initial box $[1, \infty] \times [2, 3]$. Red: contracted box.

Example of elementary contractors

Example 1: consider the constraint $a(\cdot) = x(\cdot) + y(\cdot)$
A minimal **contractor** to apply this constraint is:

$$\begin{pmatrix} [a] \\ [x] \\ [y] \end{pmatrix} \xmapsto{\mathcal{C}_+} \begin{pmatrix} [a] \cap ([x] + [y]) \\ [x] \cap ([a] - [y]) \\ [y] \cap ([a] - [x]) \end{pmatrix}$$

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Example 2: consider the constraint $y - \exp(x) = 0$
A **contractor** to apply this constraint is:

$$\begin{pmatrix} [x] \\ [y] \end{pmatrix} \xmapsto{\mathcal{C}_{\exp}} \begin{pmatrix} [x] \cap \log([y]) \\ [y] \cap \exp([x]) \end{pmatrix}$$

Contractor programming: $\mathcal{C}_{\exp}\big([x], [y]\big)$

Example of composition of contractors

Example: **decomposition** of the observation constraint:

$$\mathcal{L}_{\text{dist}} : \rho = \sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2}$$

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Example: the $\mathcal{C}_{\text{dist}}$ contractor

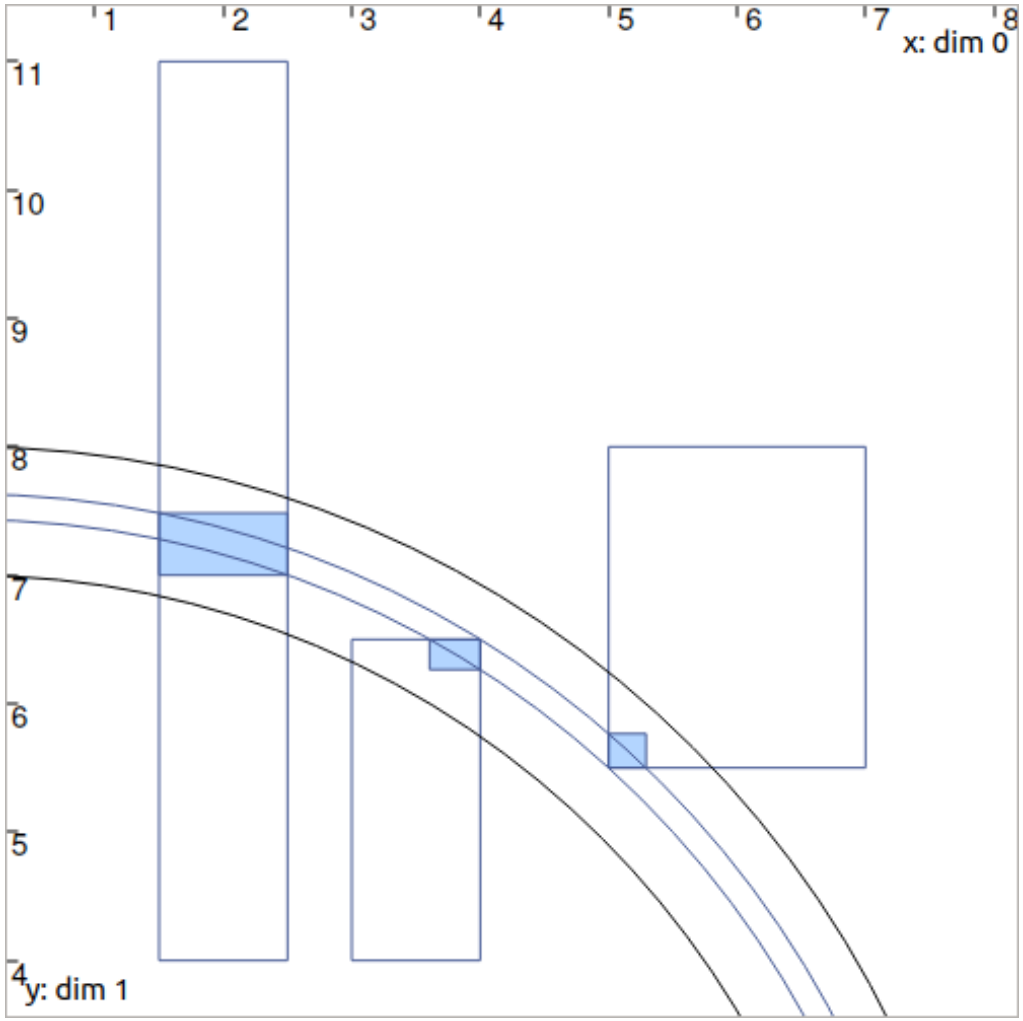

Illustration of several contracted boxes with $\mathcal{C}_{\text{dist}}$ contractor. The blue boxes have been contracted as well as the ring.

Uncertainties as sets

Example of **range-only** robot localization (three beacons):

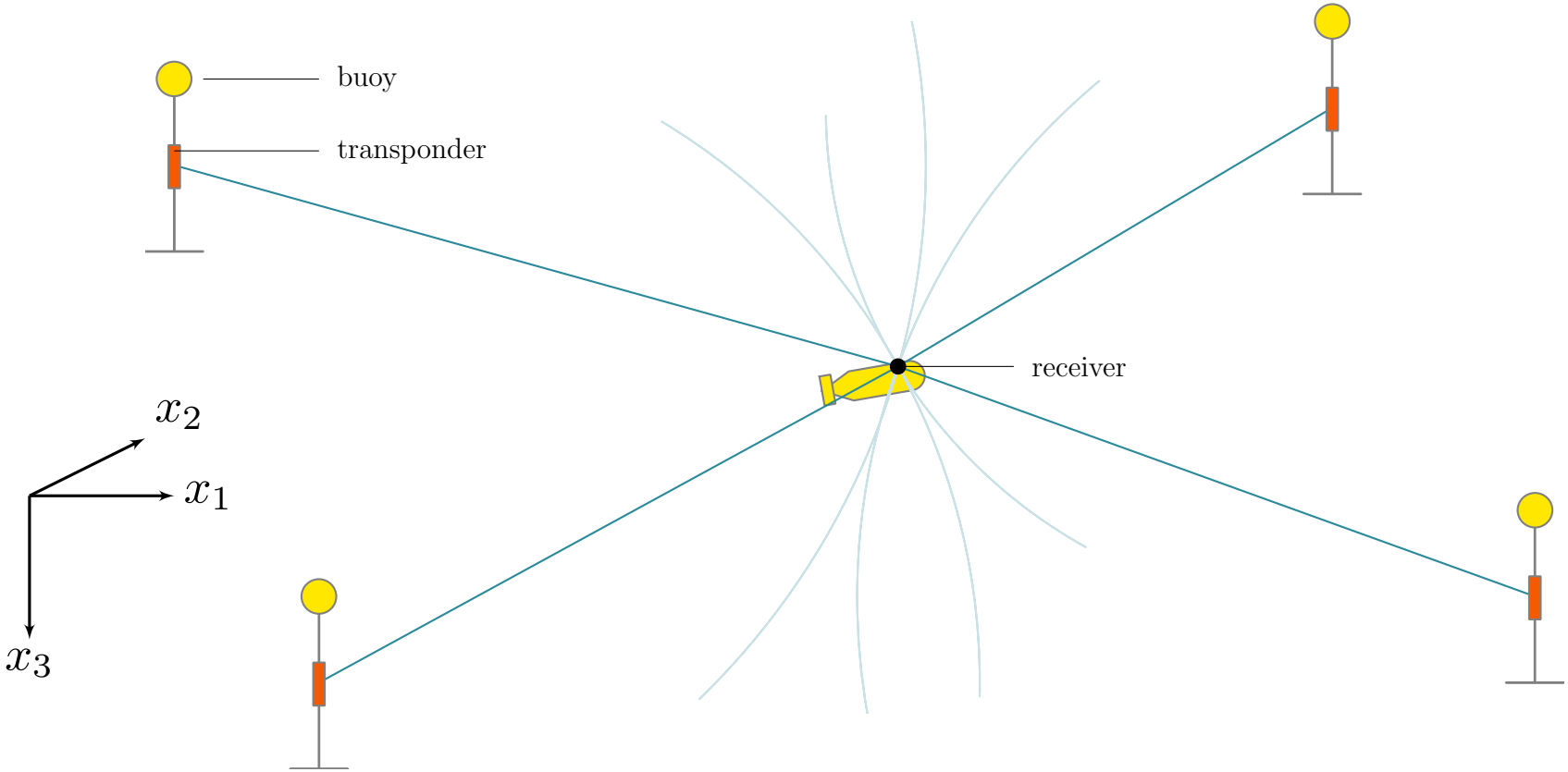

Illustration of Long BaseLine (LBL) positioning

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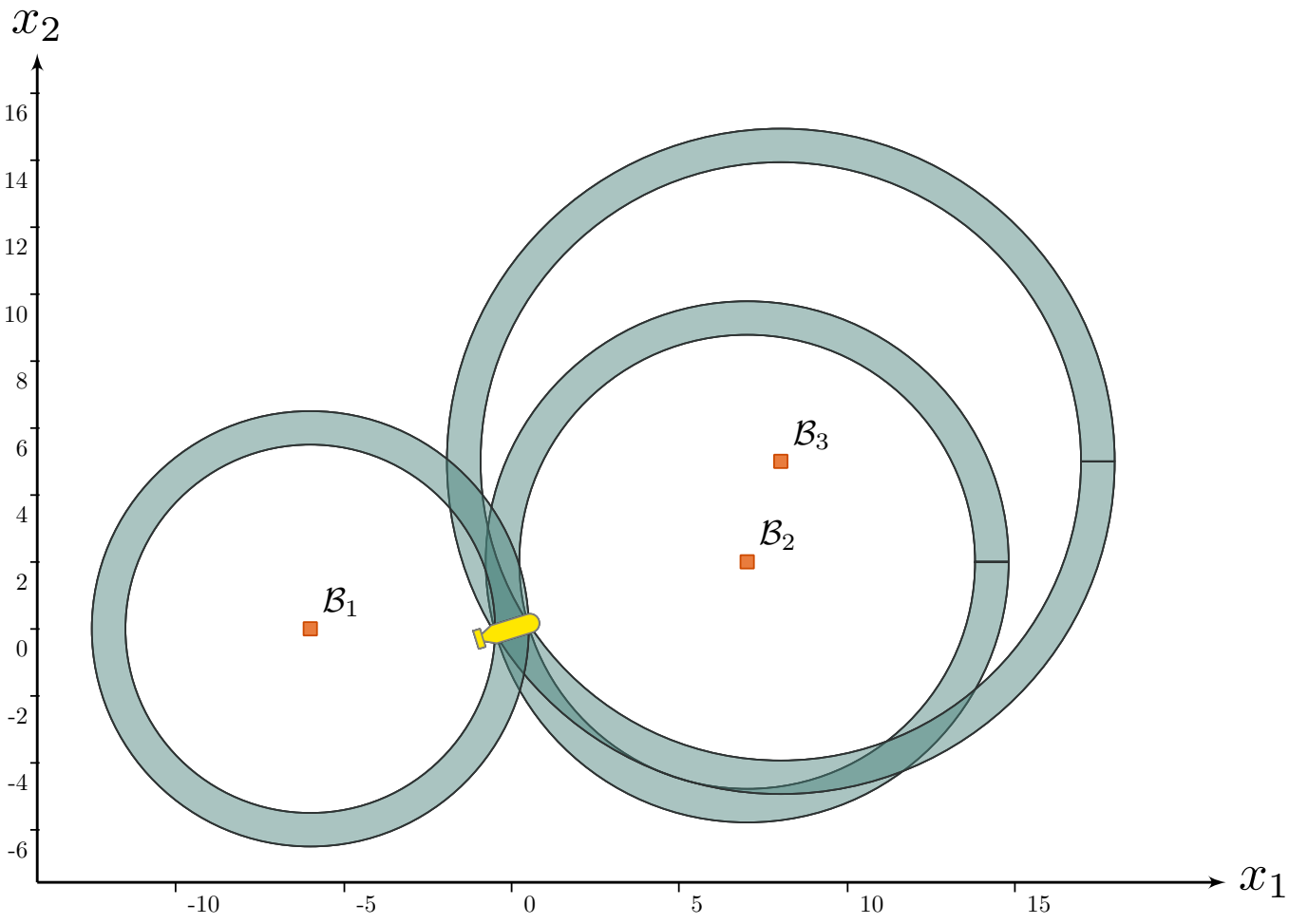
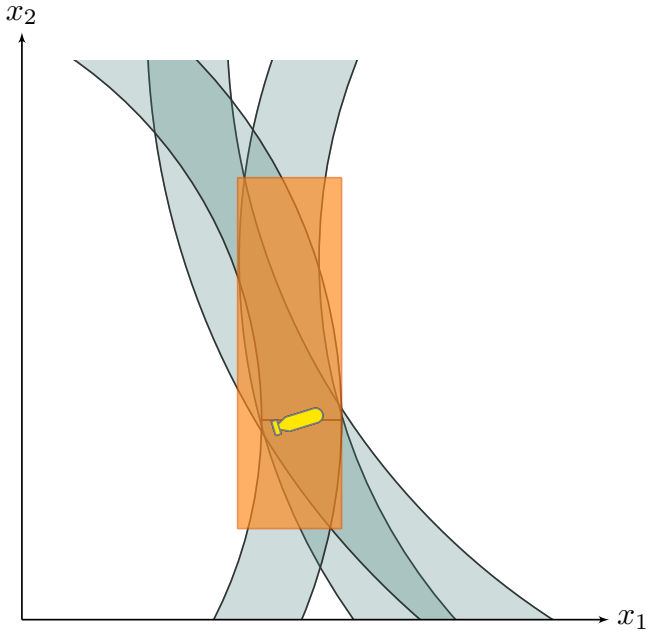


LBL positioning with bounded uncertainties

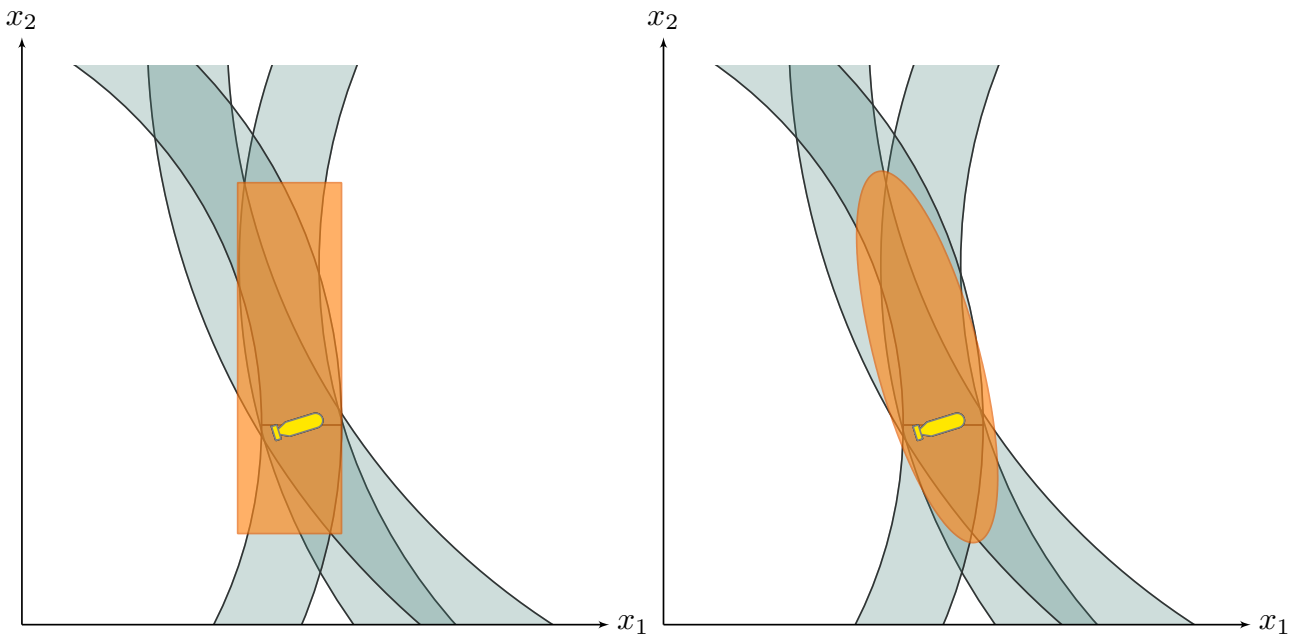
Wrappers

- box



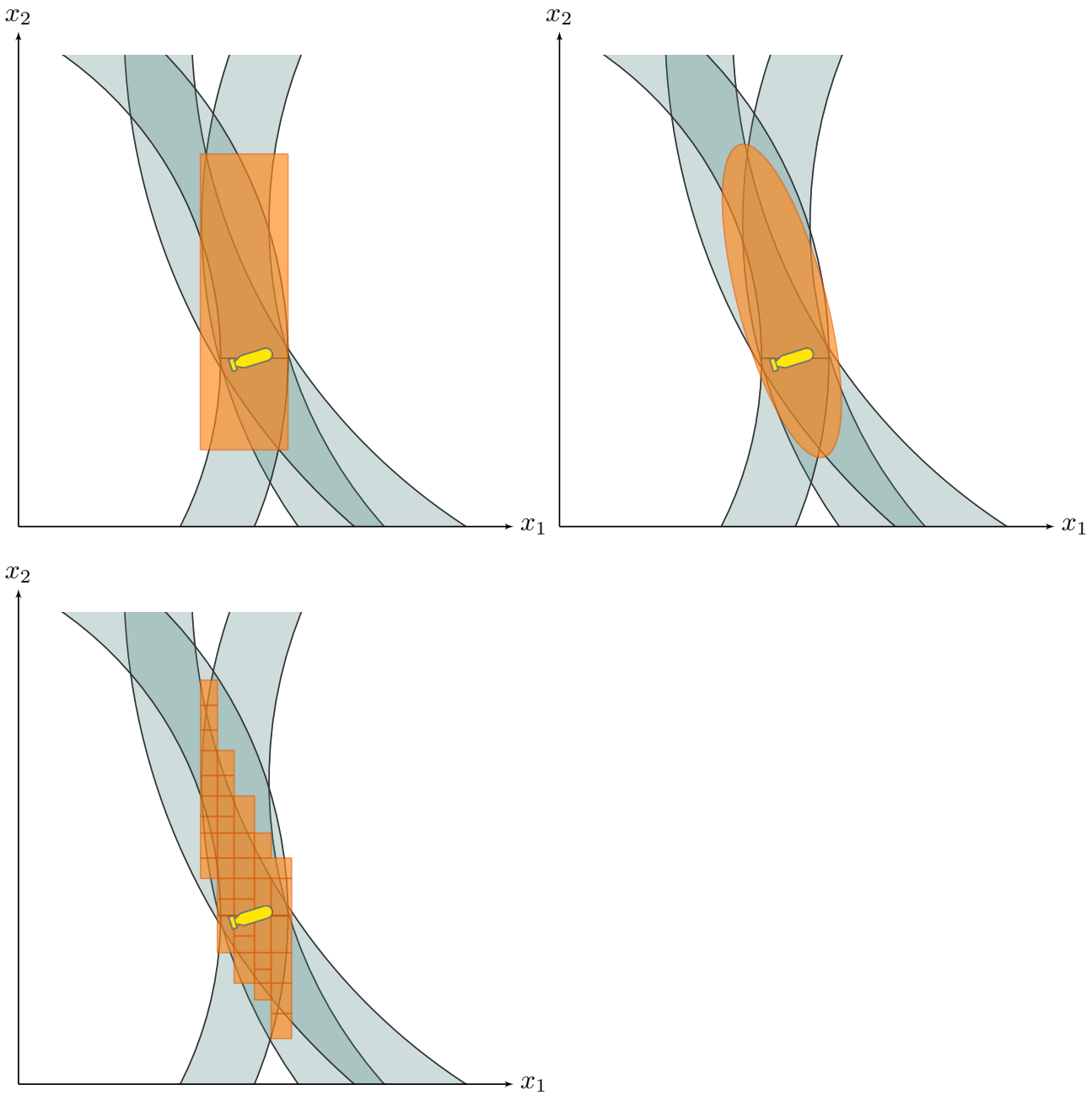
Wrappers

- box
- ellipsoid



Wrappers

- box
- ellipsoid
- paving



Wrappers

- box
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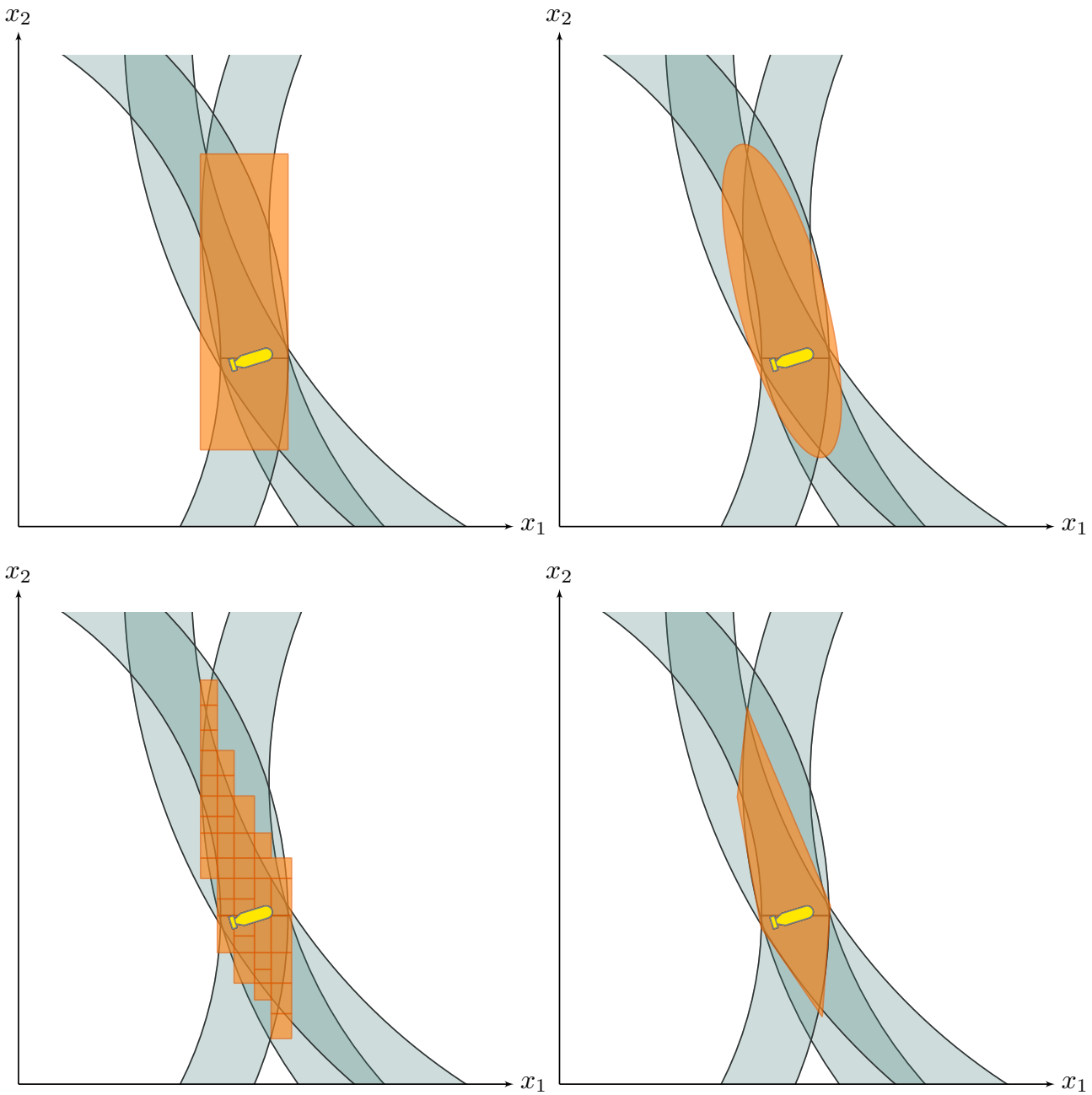
- paving
- polytopes



Wrappers

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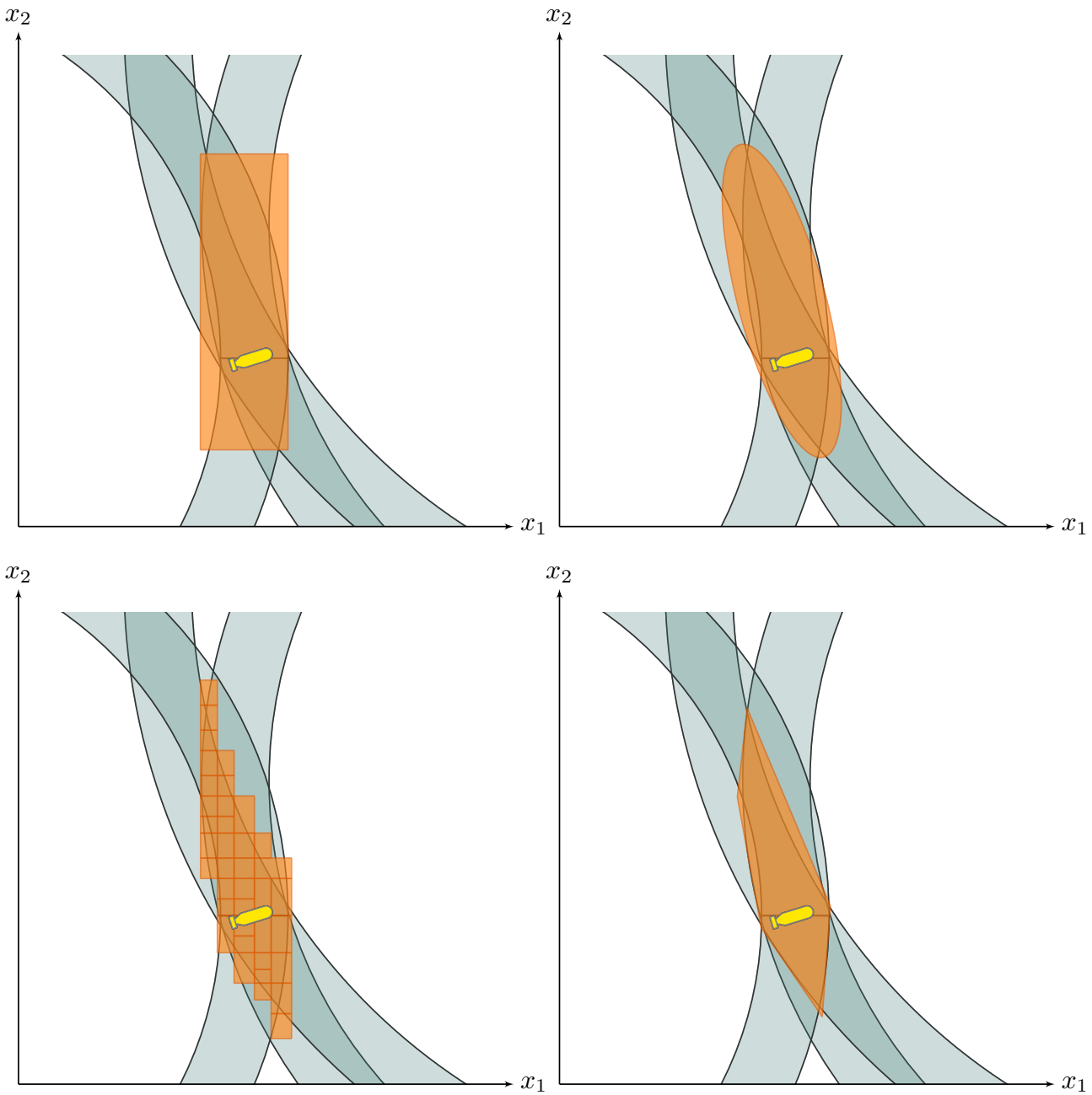
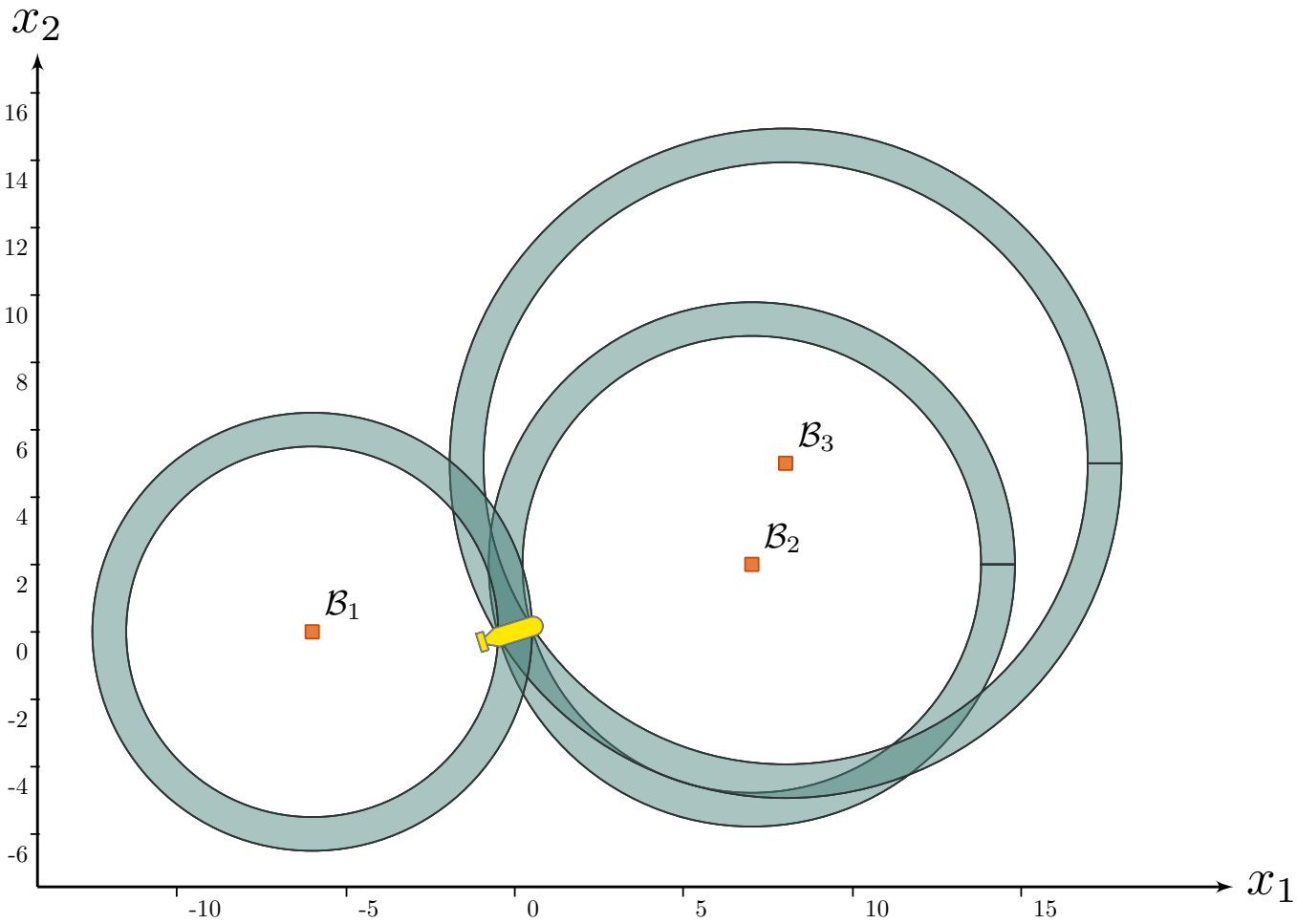


Set-membership state estimation

Three observations $\rho^{(k)}$ from three beacons $\mathcal{B}^{(k)}$:



Constraints

Observation constraint, links a measurement $\rho^{(k)}$ to the state $\mathbf{x}$:

$$\rho^{(k)} = \sqrt{\left(x_1 - \mathcal{B}_1^{(k)}\right)^2 + \left(x_2 - \mathcal{B}_2^{(k)}\right)^2}.$$

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$$\mathcal{L}_g^{(k)} : \rho^{(k)} = \sqrt{\left(x_1 - \mathcal{B}_1^{(k)}\right)^2 + \left(x_2 - \mathcal{B}_2^{(k)}\right)^2}.$$

Problem synthesized as a **constraint network**:

$$\begin{cases} \textbf{Variables: } \mathbf{x},\ \rho^{(1)},\ \rho^{(2)},\ \rho^{(3)} \\ \textbf{Constraints:} \\ \quad 1.\ \mathcal{L}_g^{(1)}\left(\mathbf{x}, \rho^{(1)}\right) \\ \quad 2.\ \mathcal{L}_g^{(2)}\left(\mathbf{x}, \rho^{(2)}\right) \\ \quad 3.\ \mathcal{L}_g^{(3)}\left(\mathbf{x}, \rho^{(3)}\right) \\ \textbf{Domains: } [\mathbf{x}],\ [\rho^{(1)}],\ [\rho^{(2)}],\ [\rho^{(3)}] \end{cases}$$

Constraints

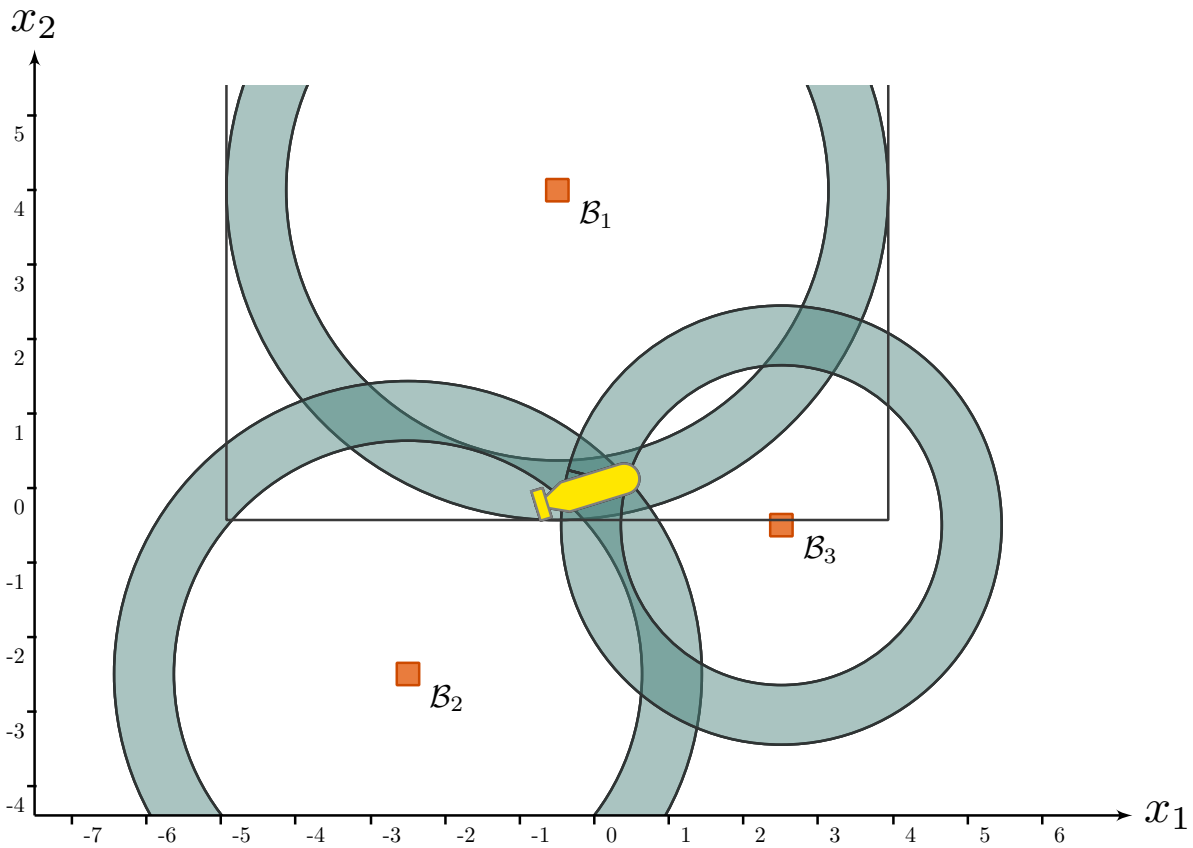
Observation constraint, links a measurement $\rho^{(k)}$ to the state $\mathbf{x}$:

$$\mathcal{L}_g^{(k)} : \rho^{(k)} = \sqrt{\left(x_1 - \mathcal{B}_1^{(k)}\right)^2 + \left(x_2 - \mathcal{B}_2^{(k)}\right)^2}.$$

Problem synthesized as a **constraint network**:

$$\begin{cases} \textbf{Variables: } \mathbf{x},\ \rho^{(1)},\ \rho^{(2)},\ \rho^{(3)} \\ \textbf{Constraints:} \\ \quad 1.\ \mathcal{L}_g^{(1)}\left(\mathbf{x}, \rho^{(1)}\right) \implies \mathcal{C}_g^{(1)}\left([\mathbf{x}], [\rho^{(1)}]\right) \\ \quad 2.\ \mathcal{L}_g^{(2)}\left(\mathbf{x}, \rho^{(2)}\right) \implies \mathcal{C}_g^{(2)}\left([\mathbf{x}], [\rho^{(2)}]\right) \\ \quad 3.\ \mathcal{L}_g^{(3)}\left(\mathbf{x}, \rho^{(3)}\right) \implies \mathcal{C}_g^{(3)}\left([\mathbf{x}], [\rho^{(3)}]\right) \\ \textbf{Domains: } [\mathbf{x}],\ [\rho^{(1)}],\ [\rho^{(2)}],\ [\rho^{(3)}] \end{cases}$$

Fixed point propagations

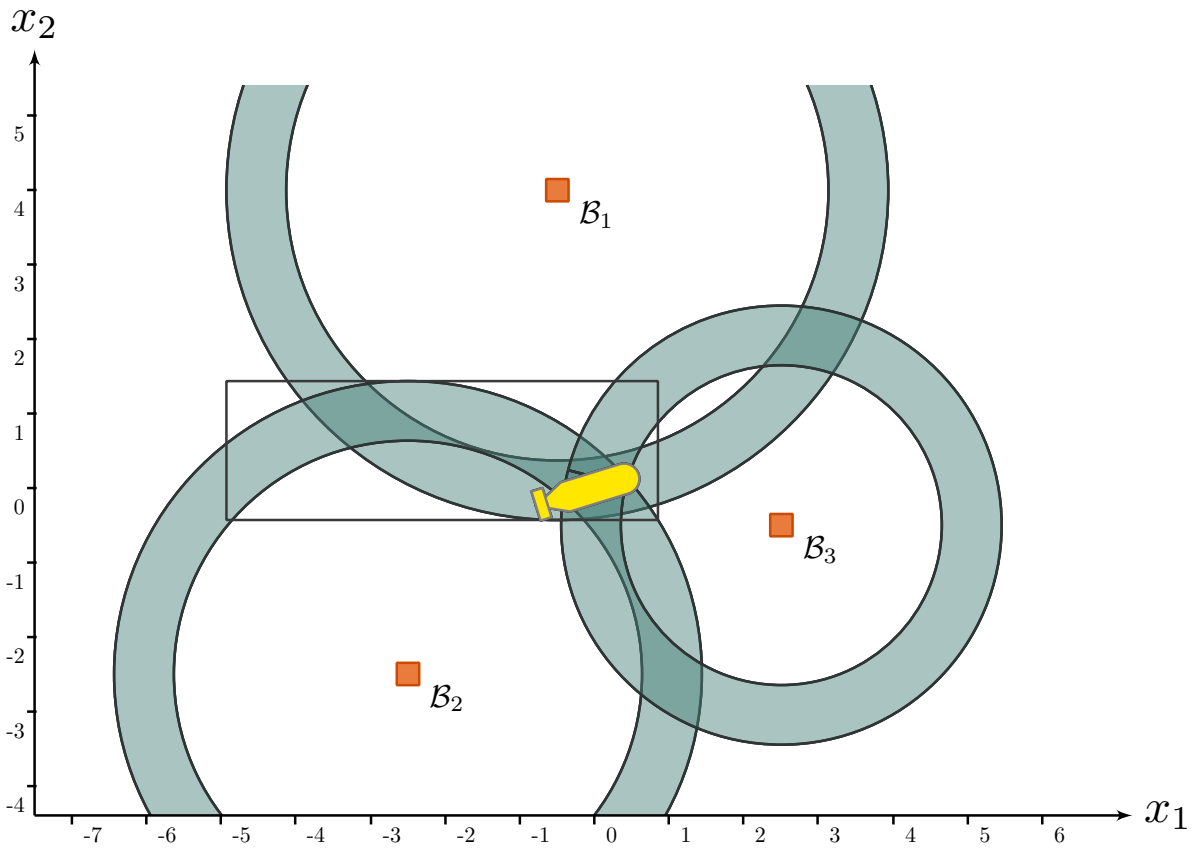


Applying $\mathcal{C}_g^{(1)}$

- Study of robust set estimation methods for a high integrity multi-sensor localization.

Vincent Drevelle *Thesis*, 2011

Fixed point propagations

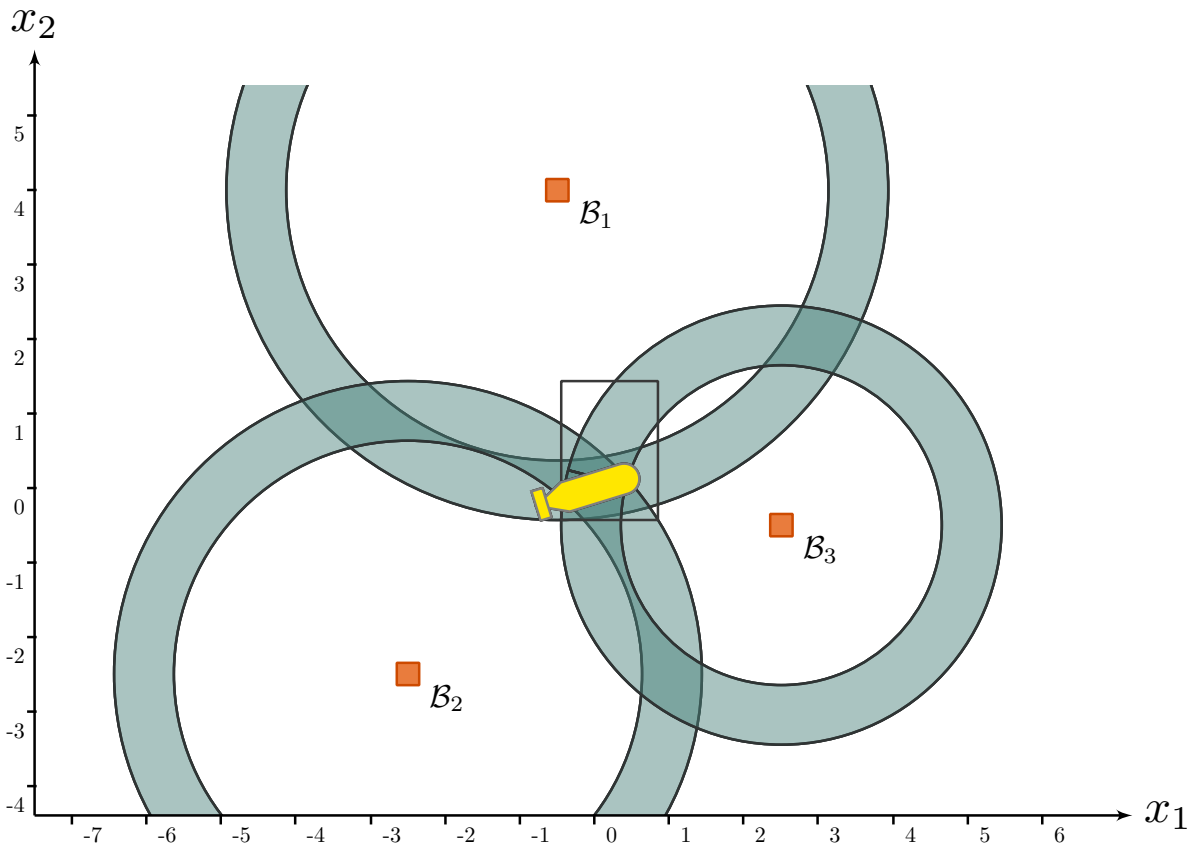


Applying $\mathcal{C}_g^{(2)}$

- Study of robust set estimation methods for a high integrity multi-sensor localization.

Vincent Drevelle *Thesis*, 2011

Fixed point propagations

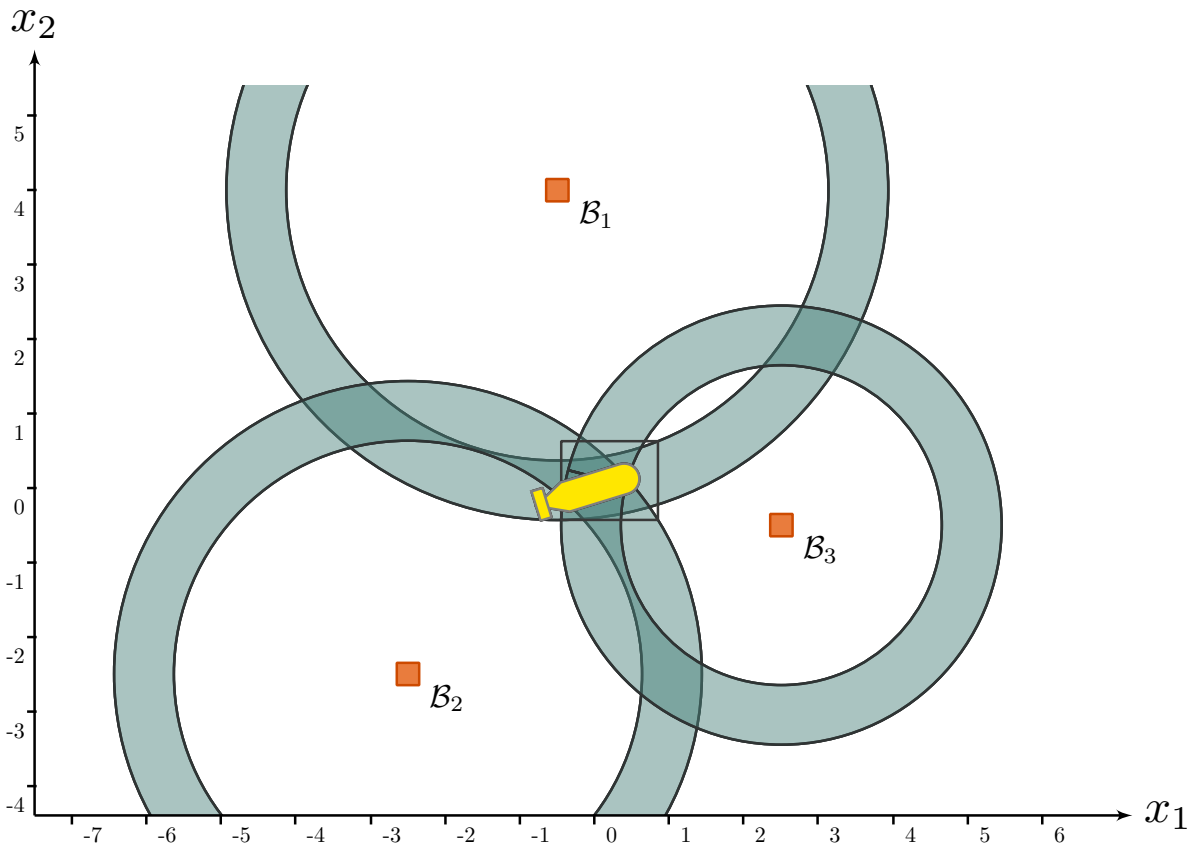


Applying $\mathcal{C}_g^{(3)}$

- Study of robust set estimation methods for a high integrity multi-sensor localization.

Vincent Drevelle *Thesis*, 2011

Fixed point propagations



Applying $\mathcal{C}_g^{(1)}$

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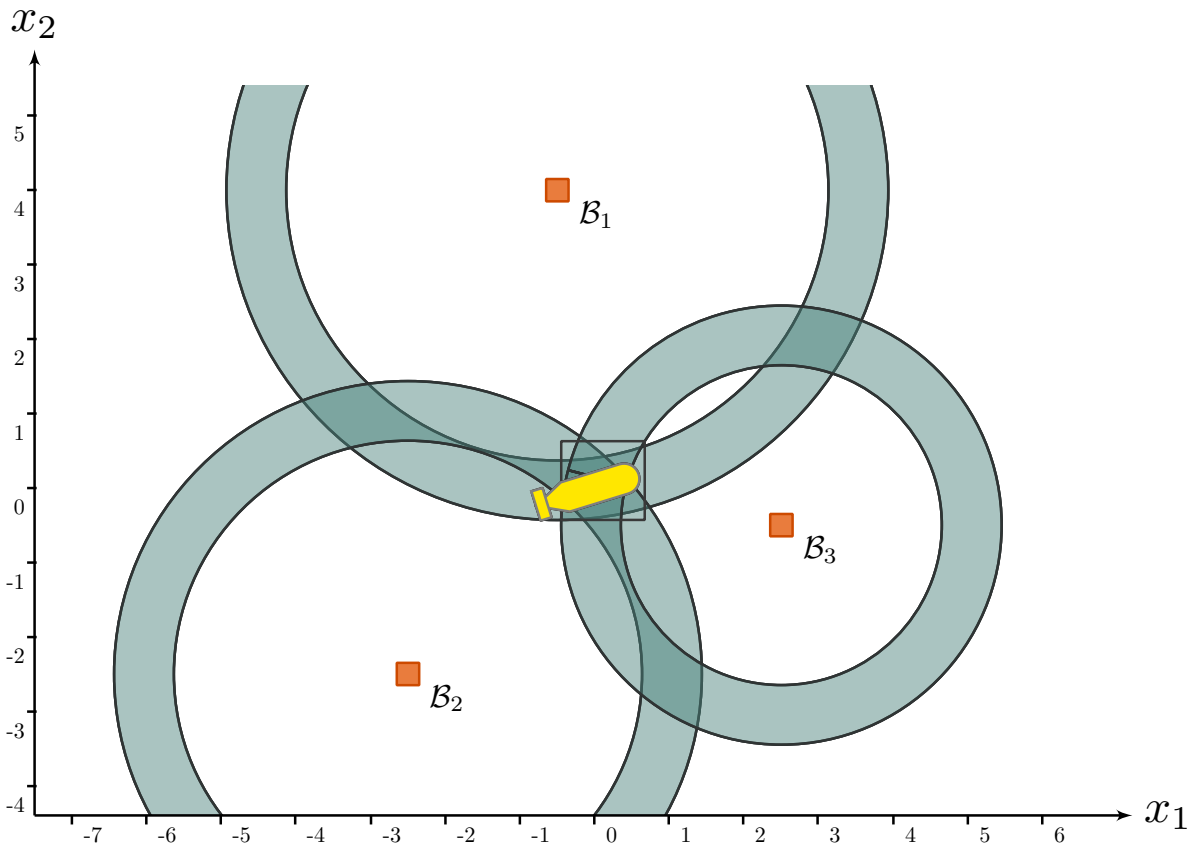
Vincent Drevelle *Thesis*, 2011

Fixed point propagations



Applying $\mathcal{C}_g^{(2)}$

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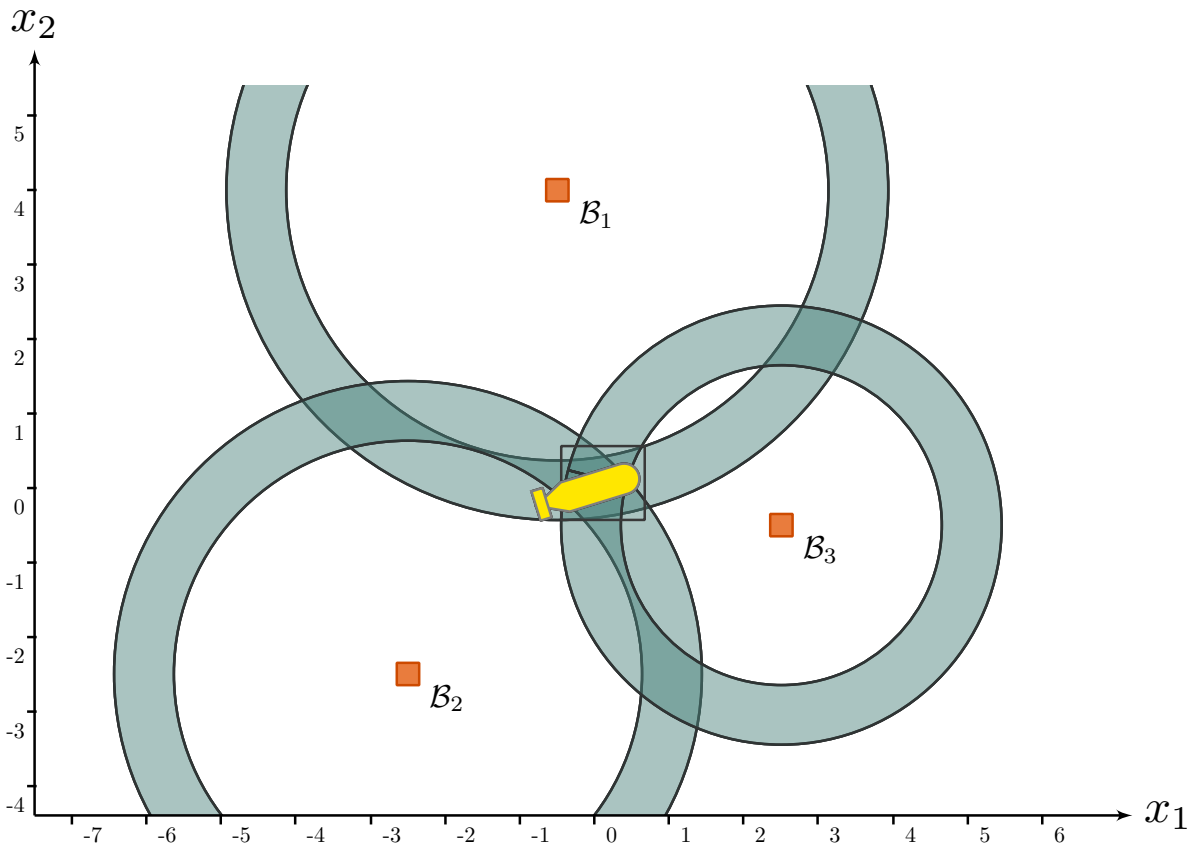
Vincent Drevelle *Thesis*, 2011

Fixed point propagations

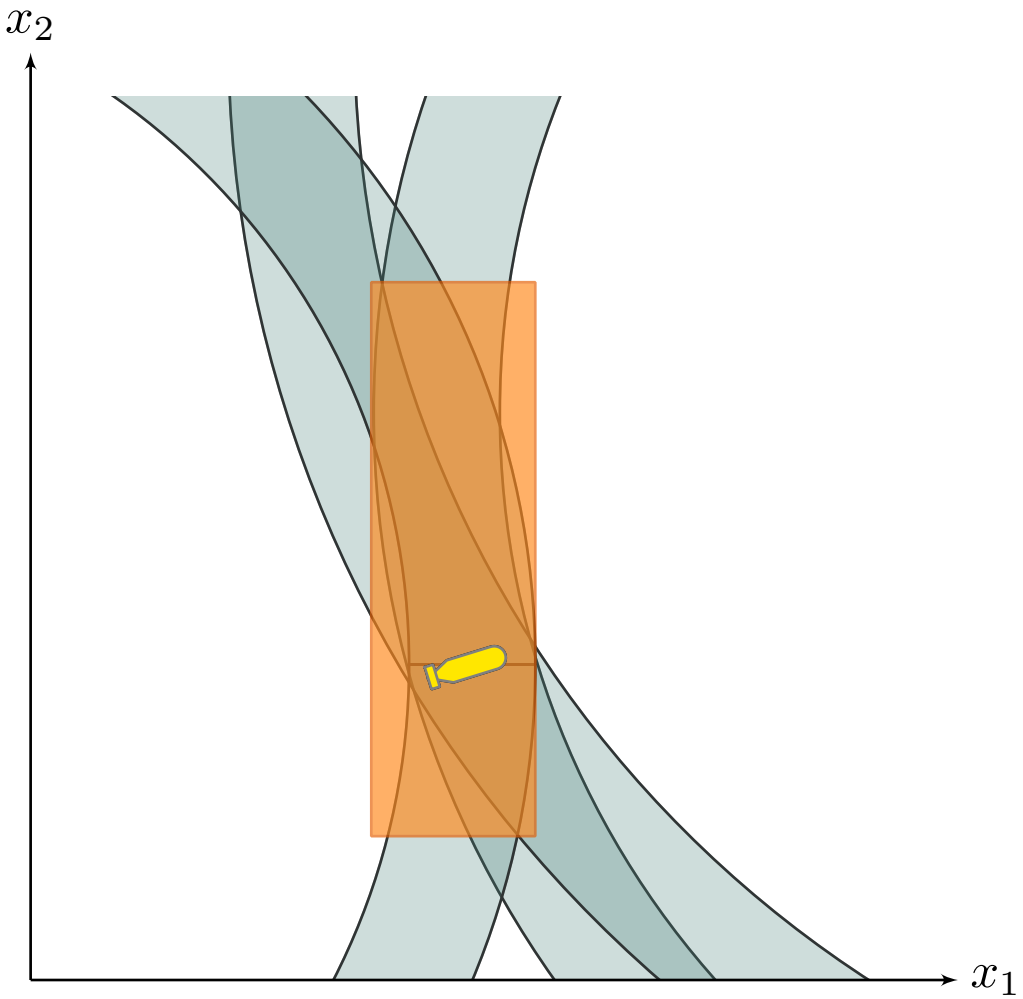


Fixed point reached.

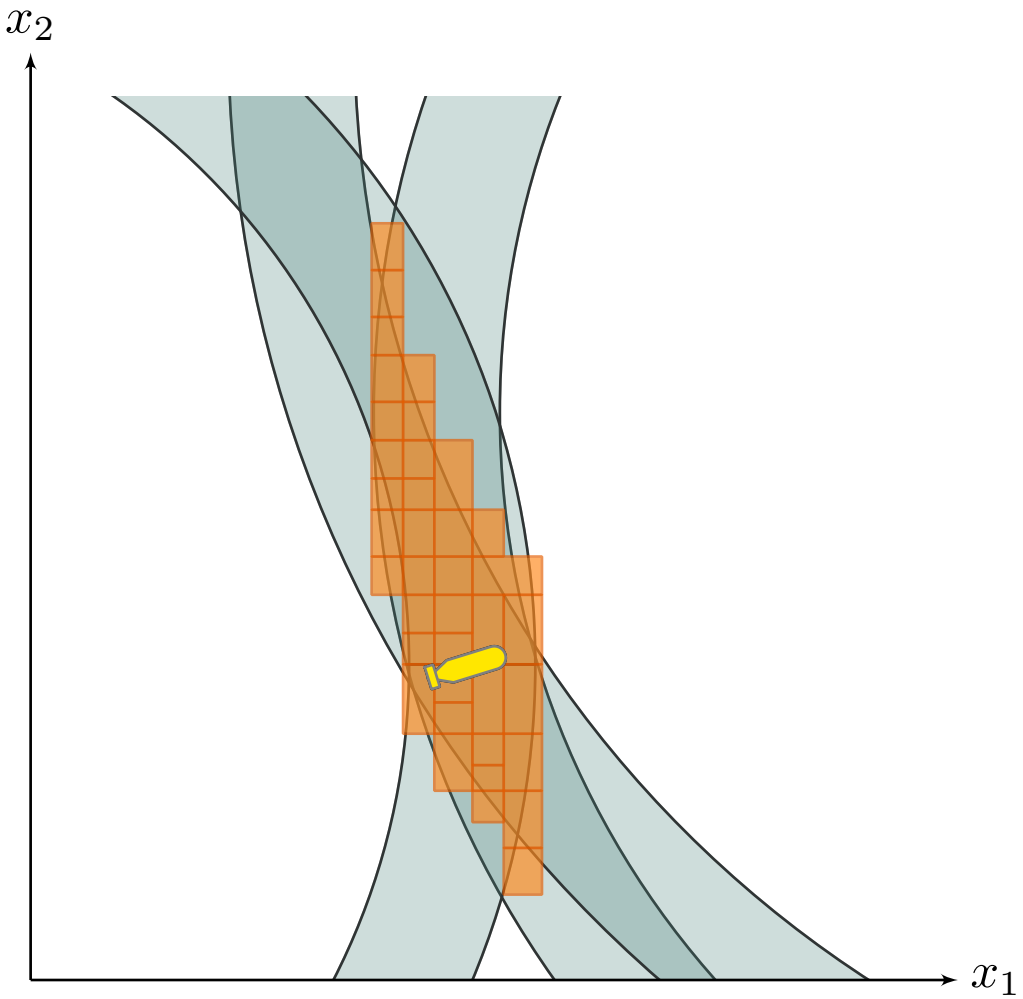
- Study of robust set estimation methods for a high integrity multi-sensor localization.

Vincent Drevelle *Thesis*, 2011

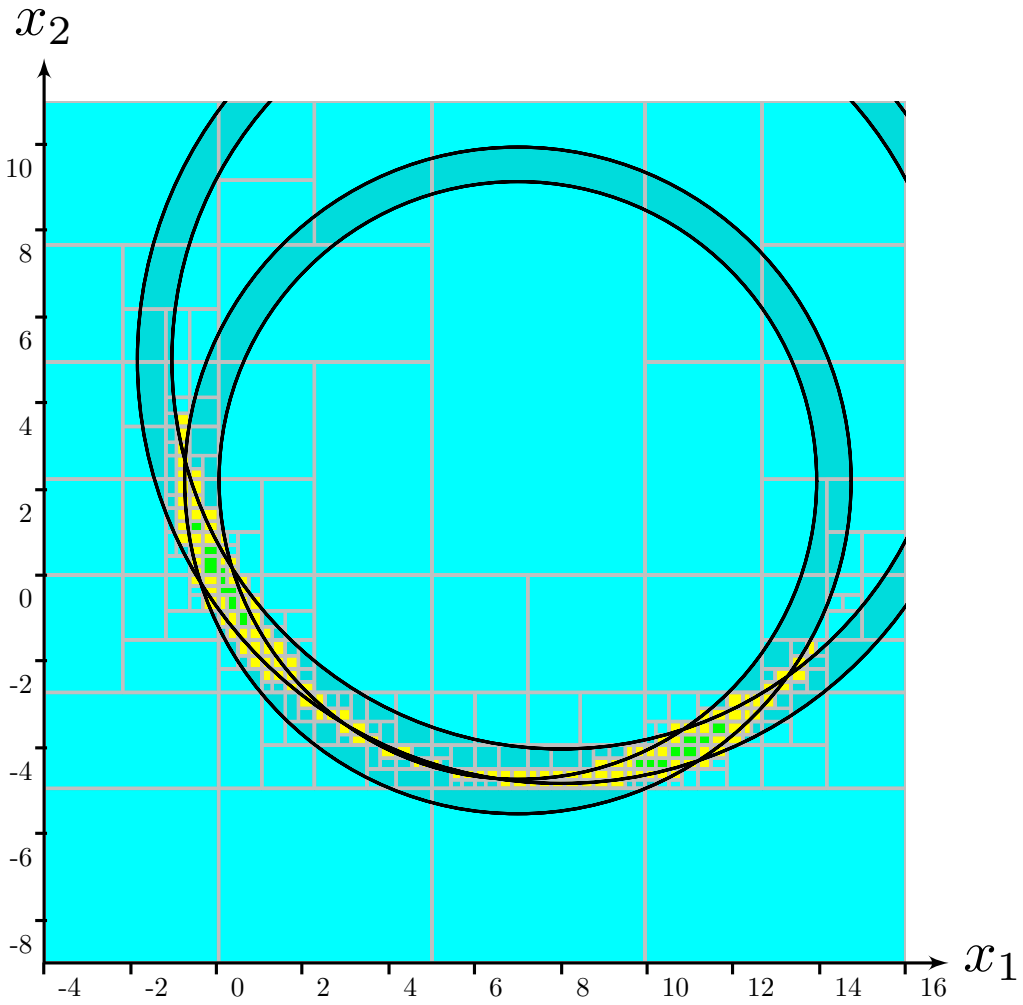
Sub-pavings: finer approximation of sets



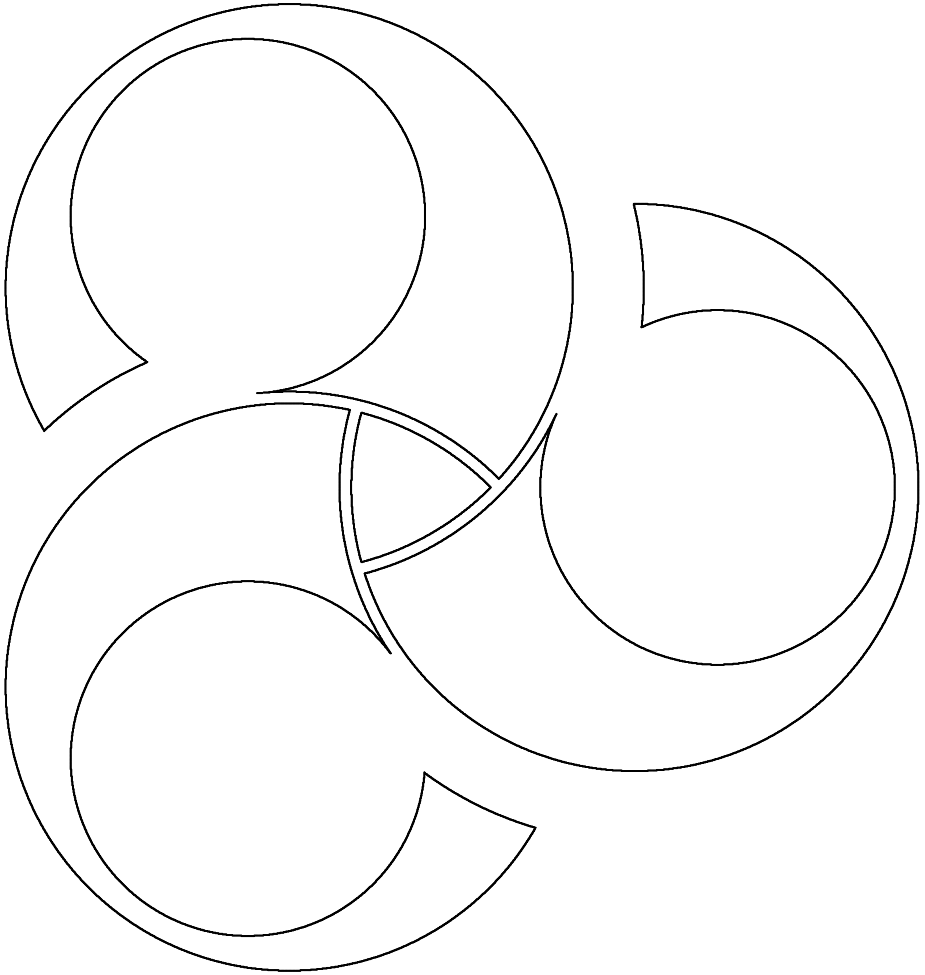
Sub-pavings: finer approximation of sets



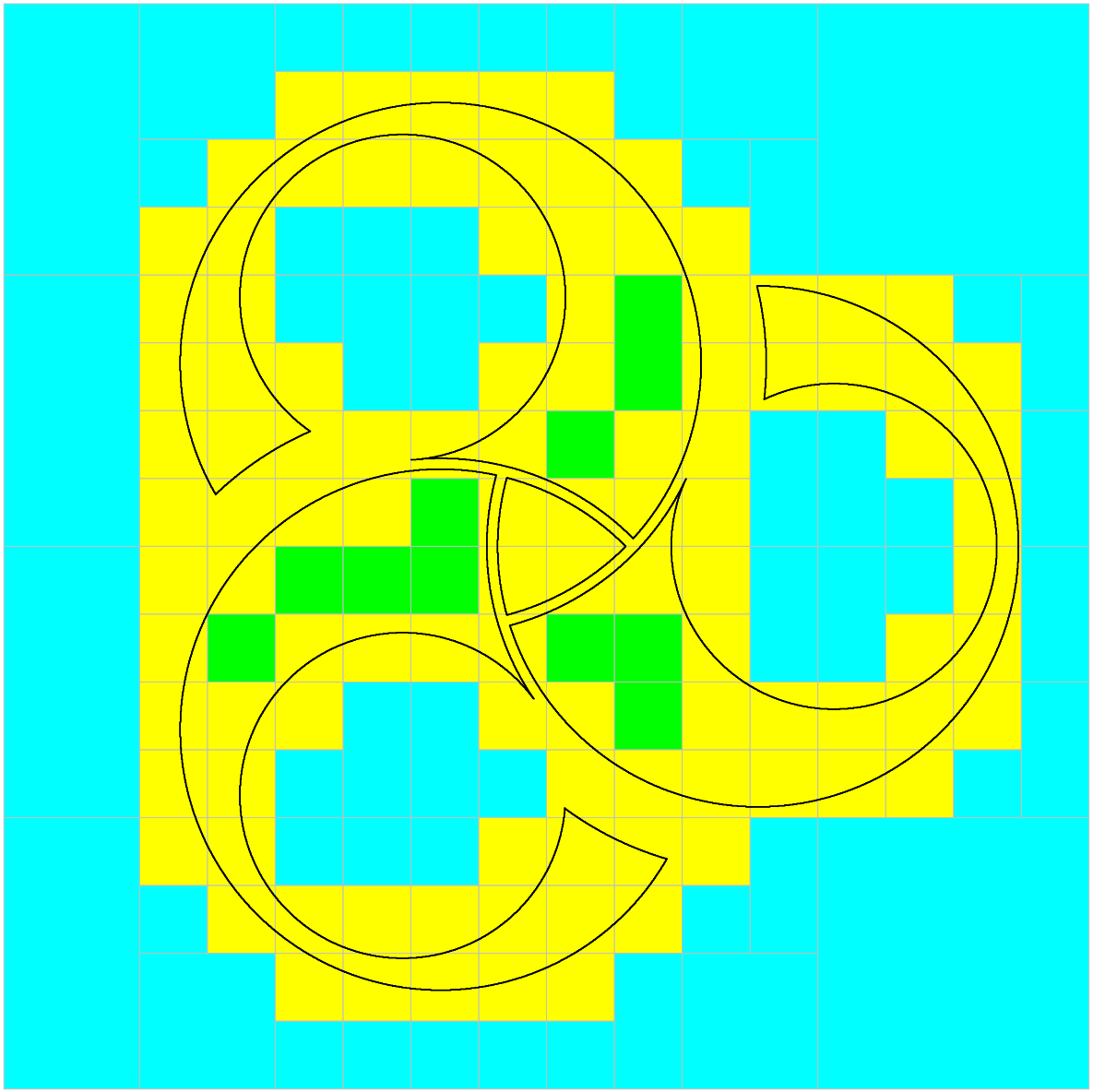
Sub-pavings: finer approximation of sets



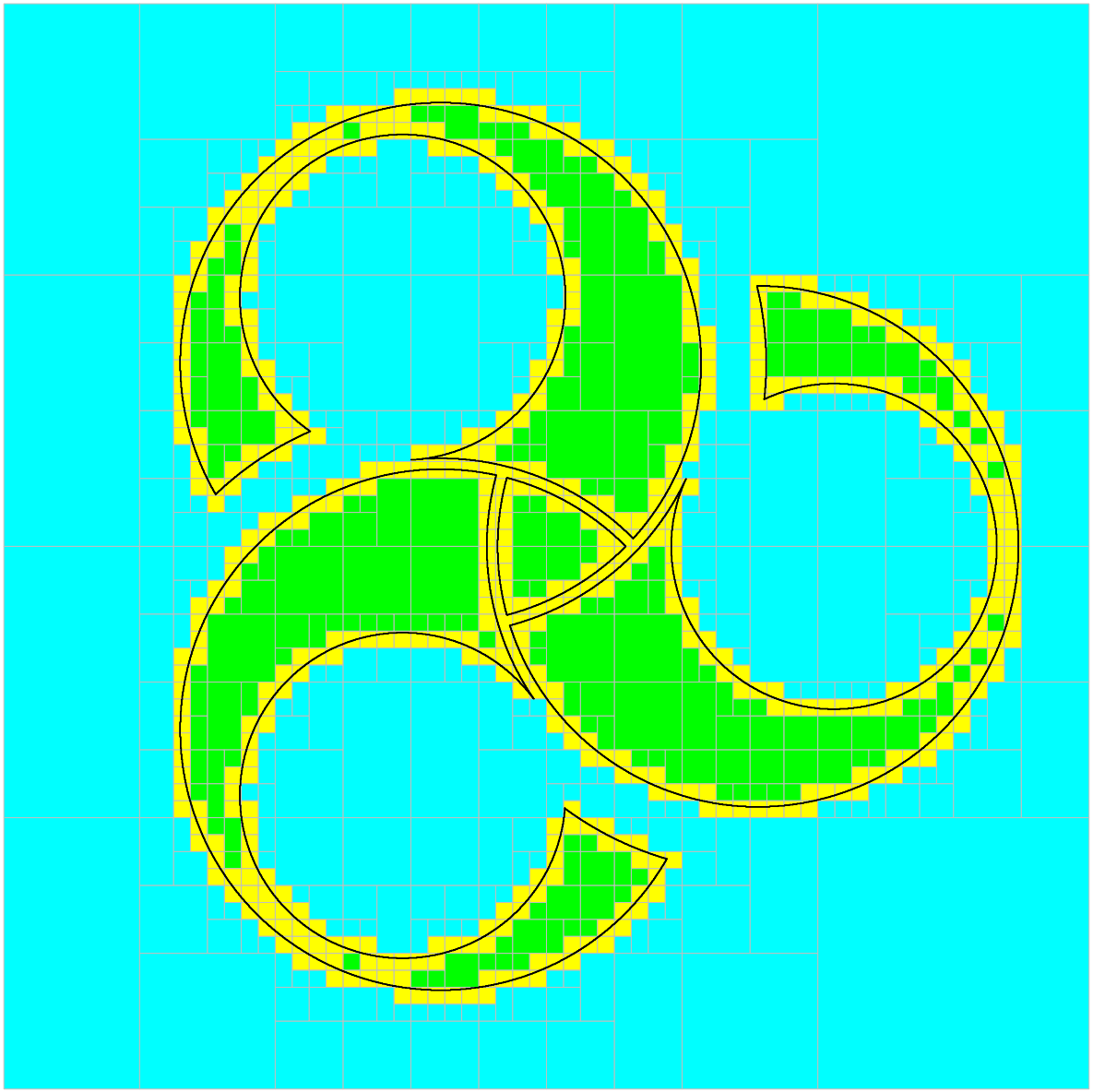
Sub-pavings: precision



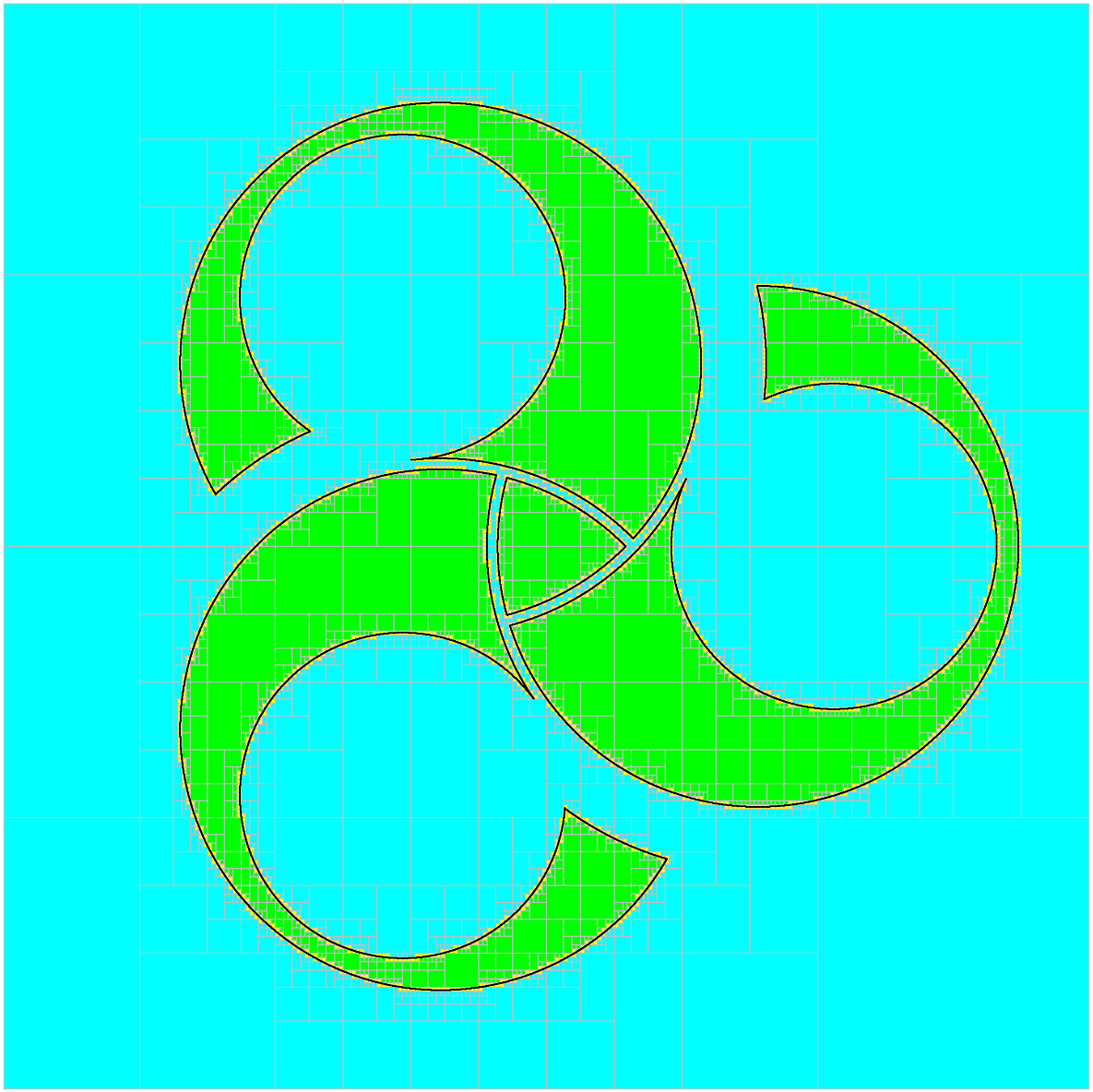
Sub-pavings: precision



Sub-pavings: precision



Sub-pavings: precision



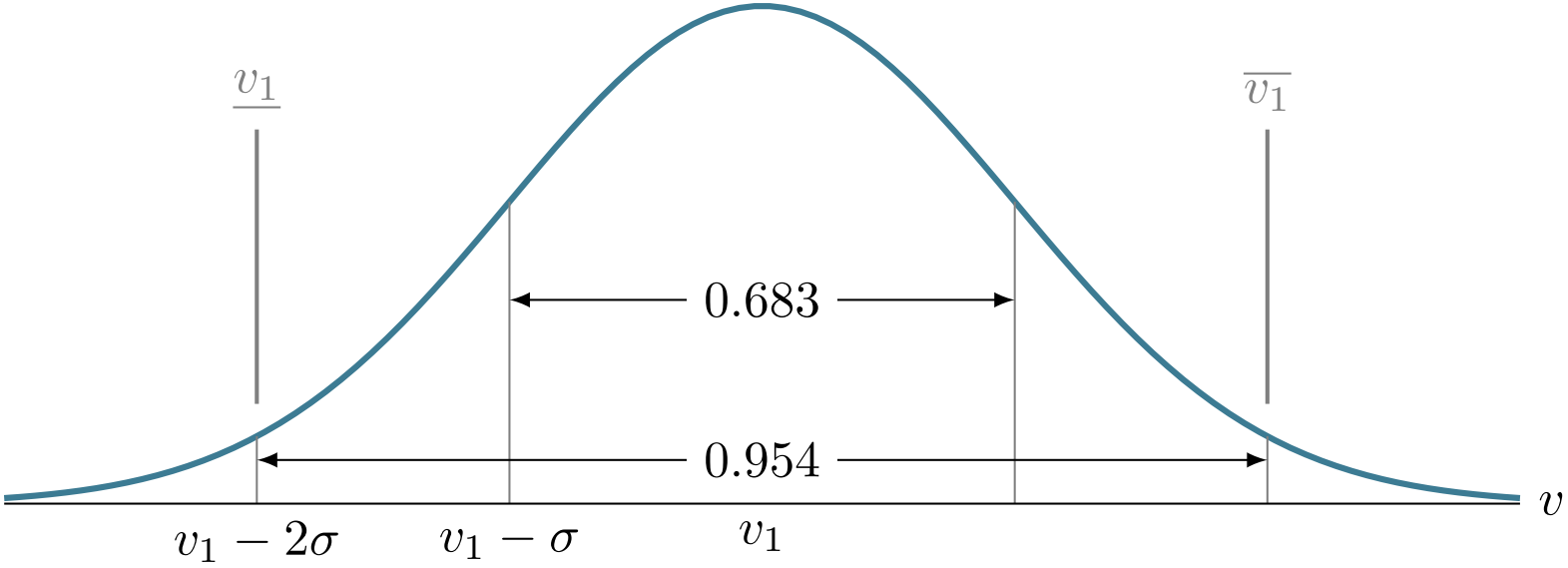
Intervals from sensor data

Video

Intervals from sensor data

The Gaussian case:

- datasheets $\implies$ standard deviation σ for each sensor
- 95% confidence rate: $v_1^* \in [v_1] = [v_1 - 2\sigma, v_1 + 2\sigma]$



Intervals from sensor data

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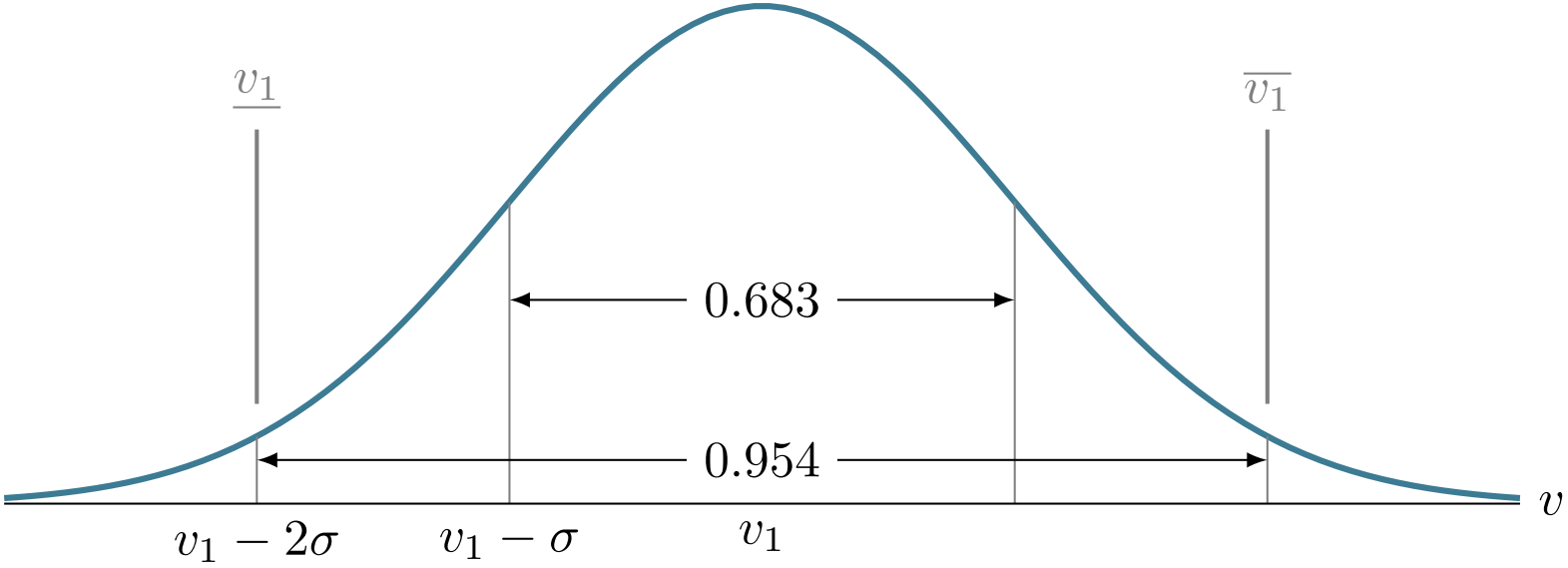


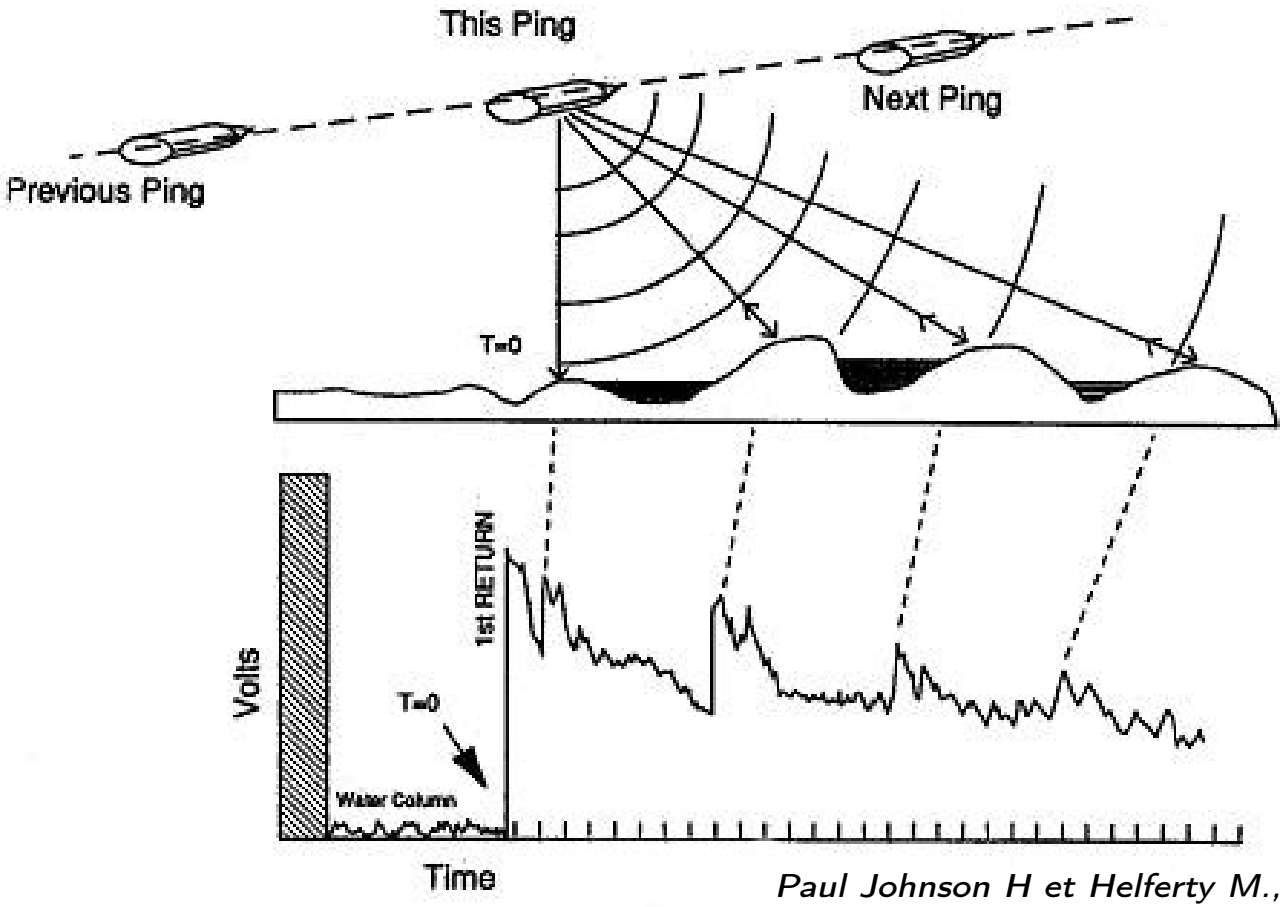
- uncertainties then reliably propagated in the system

  ex: $[x] + [y] = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right]$

Intervals from sensor data

The realistic case:

- the error is rarely a Gaussian distribution
- multimodal distribution, "no-signal" information, *etc.*



Paul Johnson H et Helferty M., 1990

Intervals from sensor data

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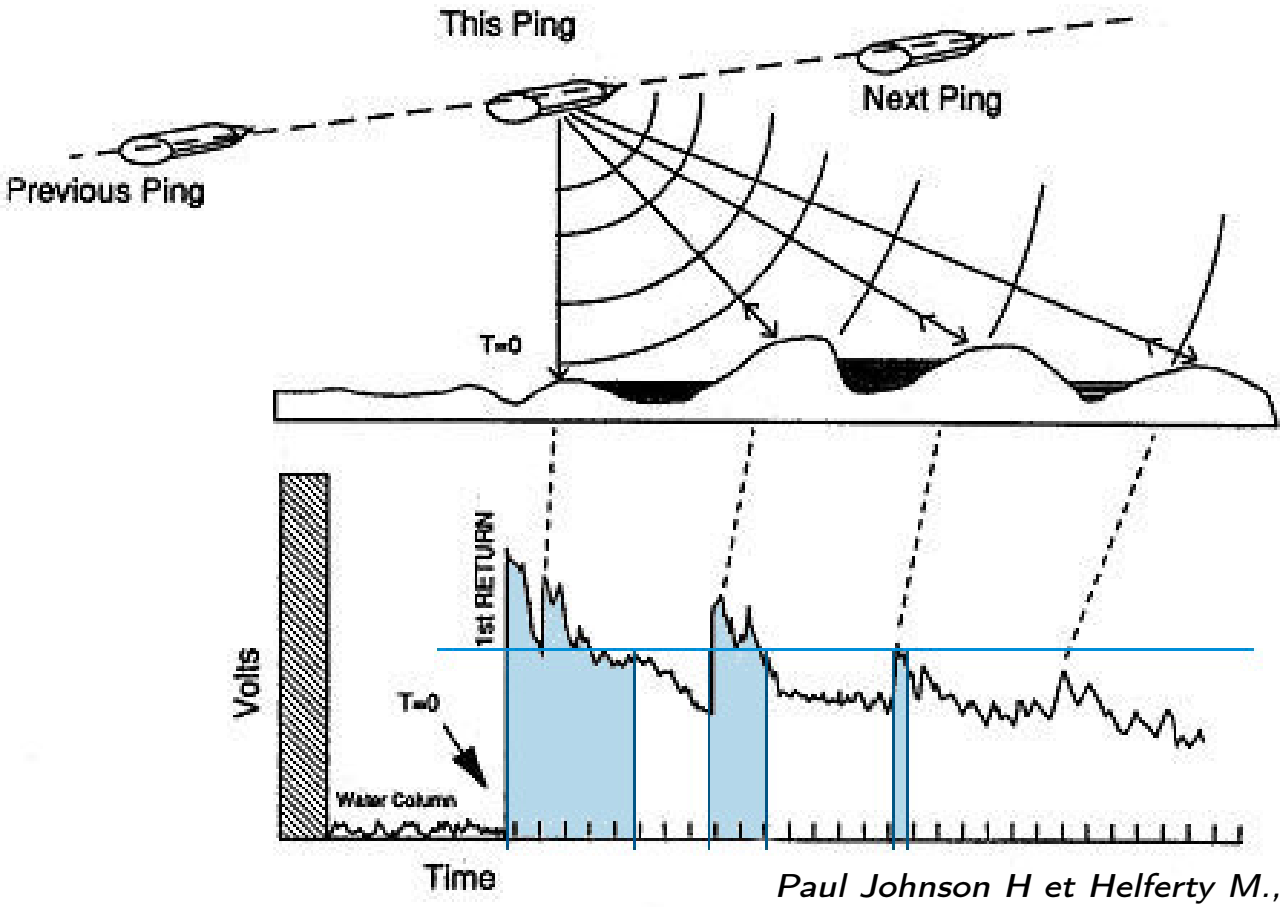

Paul Johnson H et Helferty M., 1990

Intervals from sensor data

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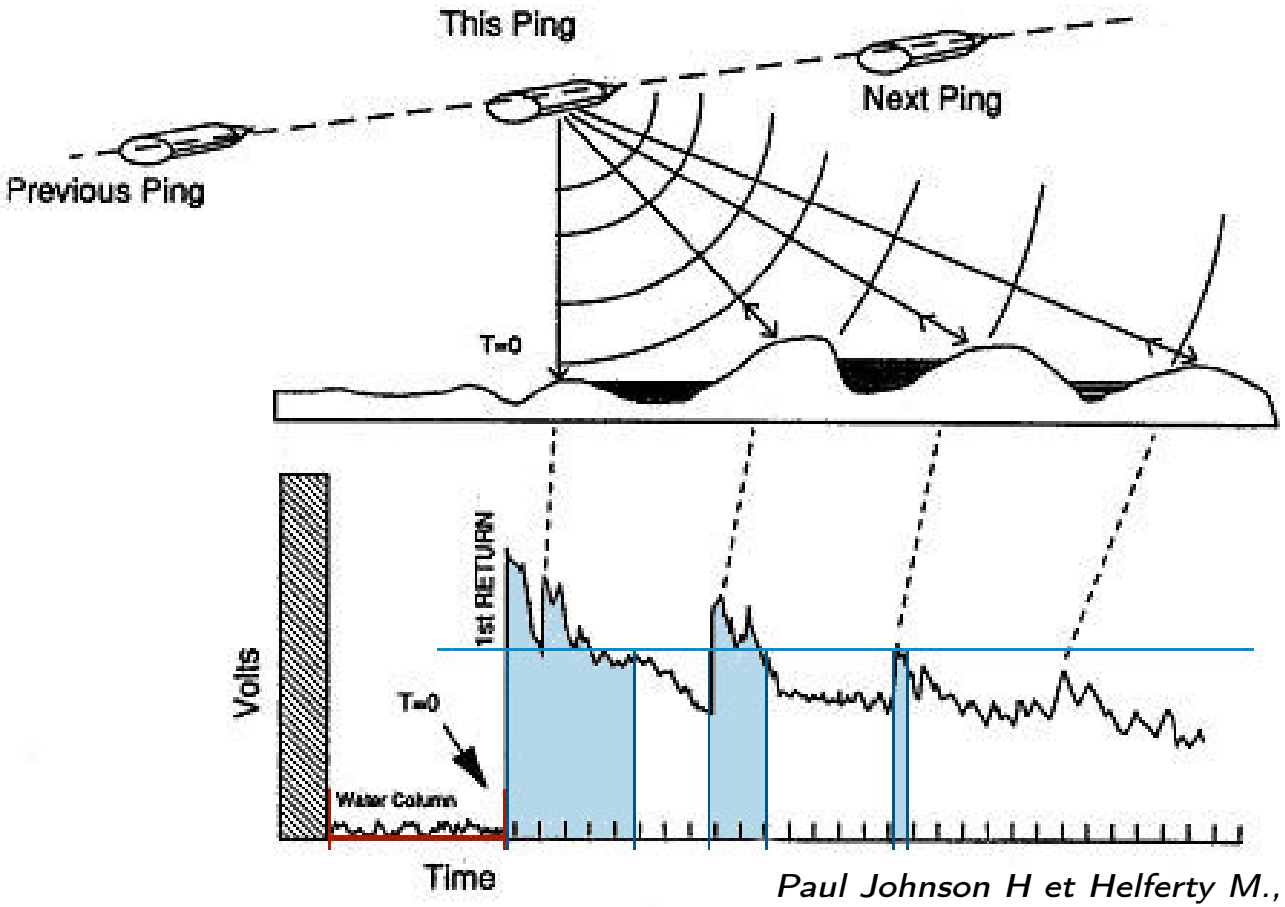
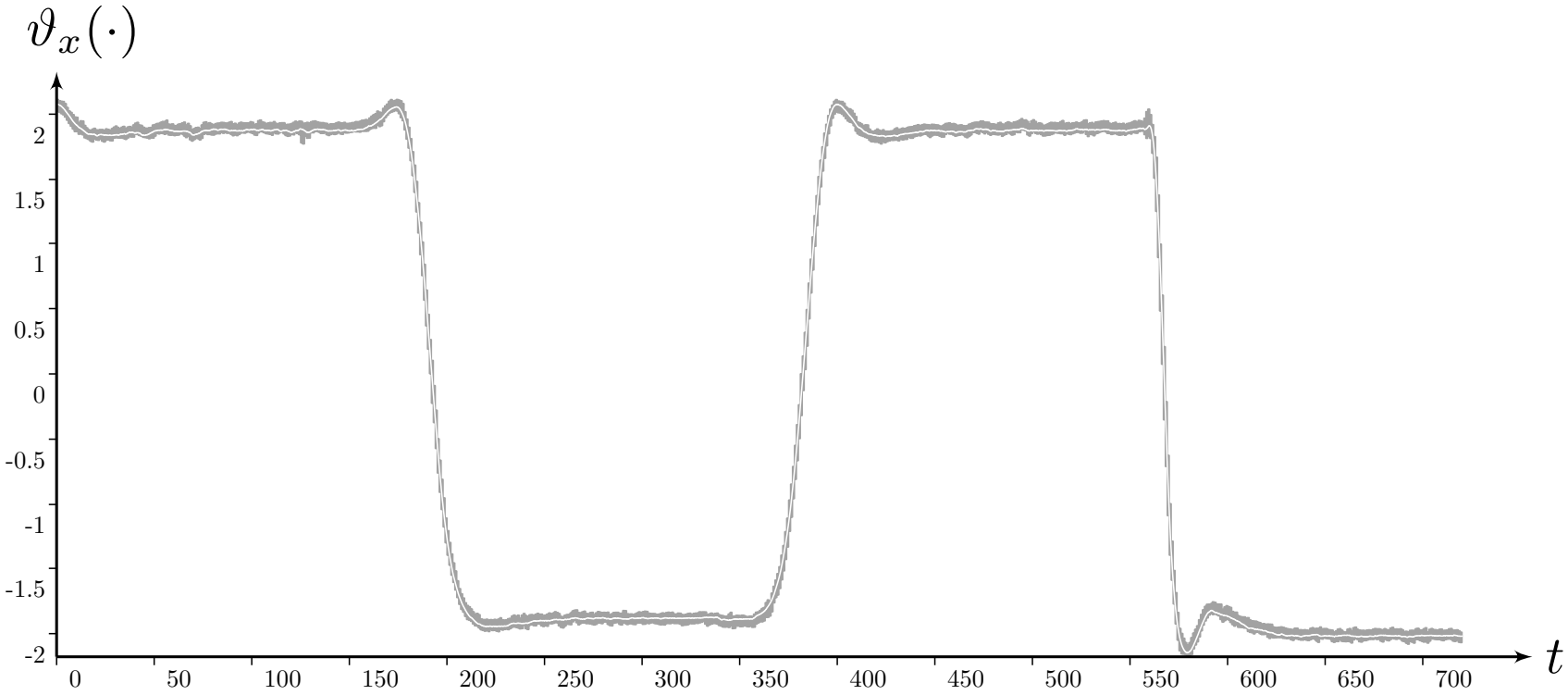


Paul Johnson H et Helferty M., 1990

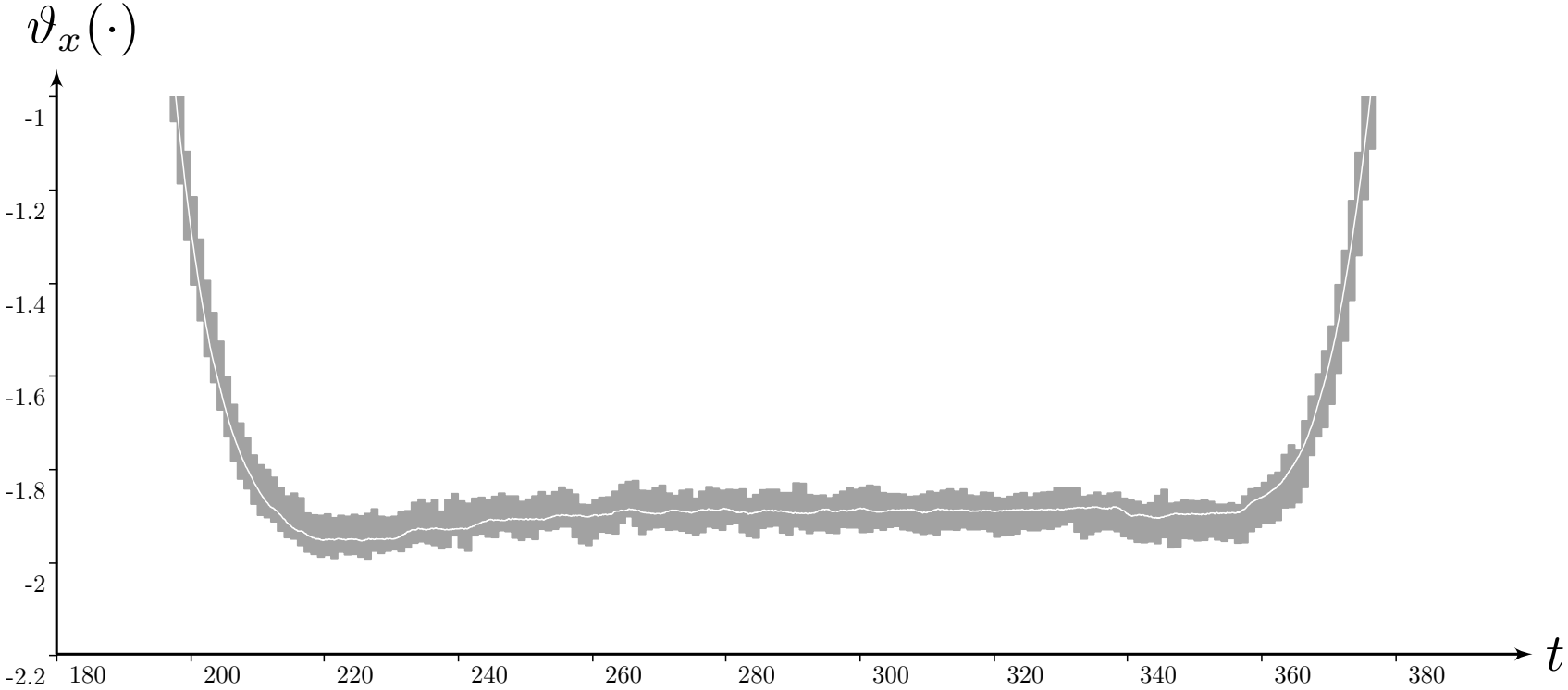
Tubes from sensor data

Example coming from underwater robotics.
East velocity given by DVL + IMU:



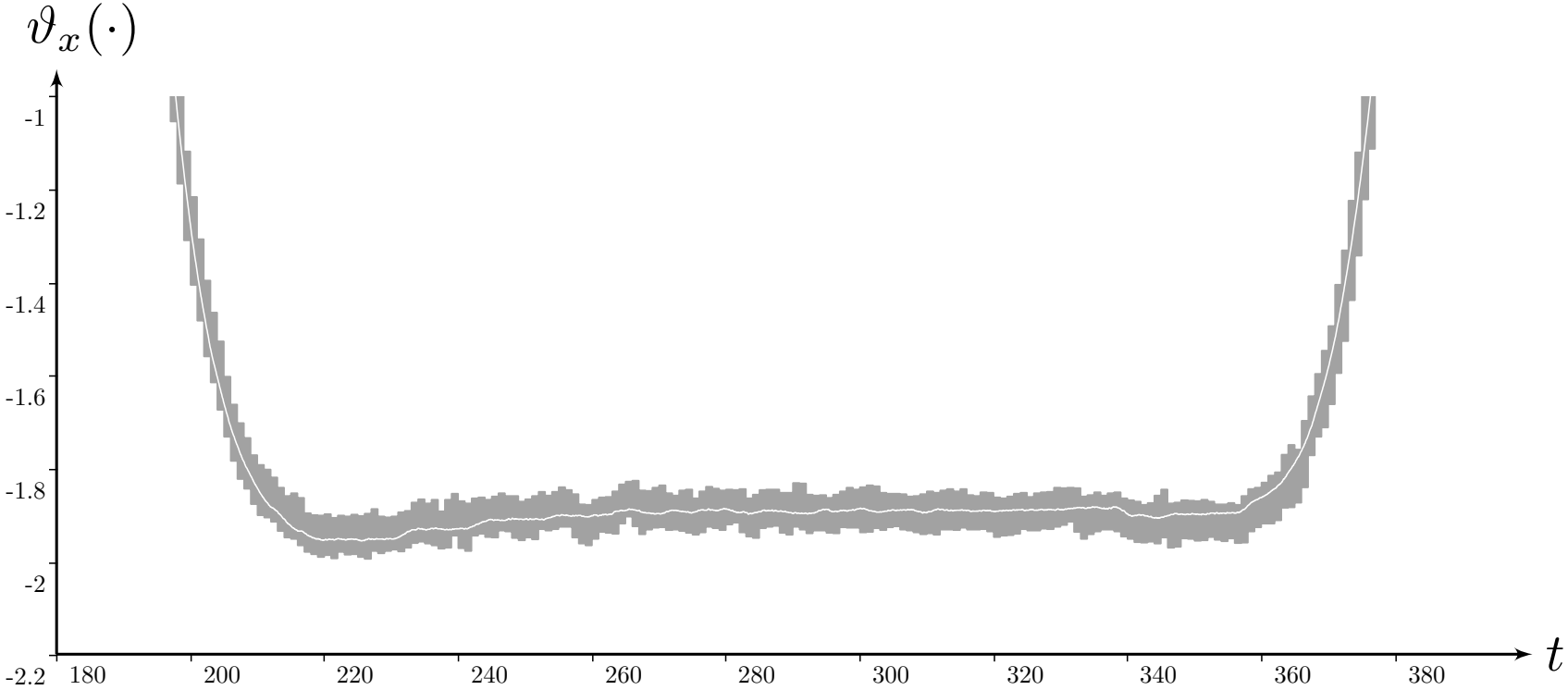
Tubes from sensor data

Example coming from underwater robotics.
East velocity given by DVL + IMU (zoom):



Tubes from sensor data

Example coming from underwater robotics.
East velocity given by DVL + IMU (zoom):

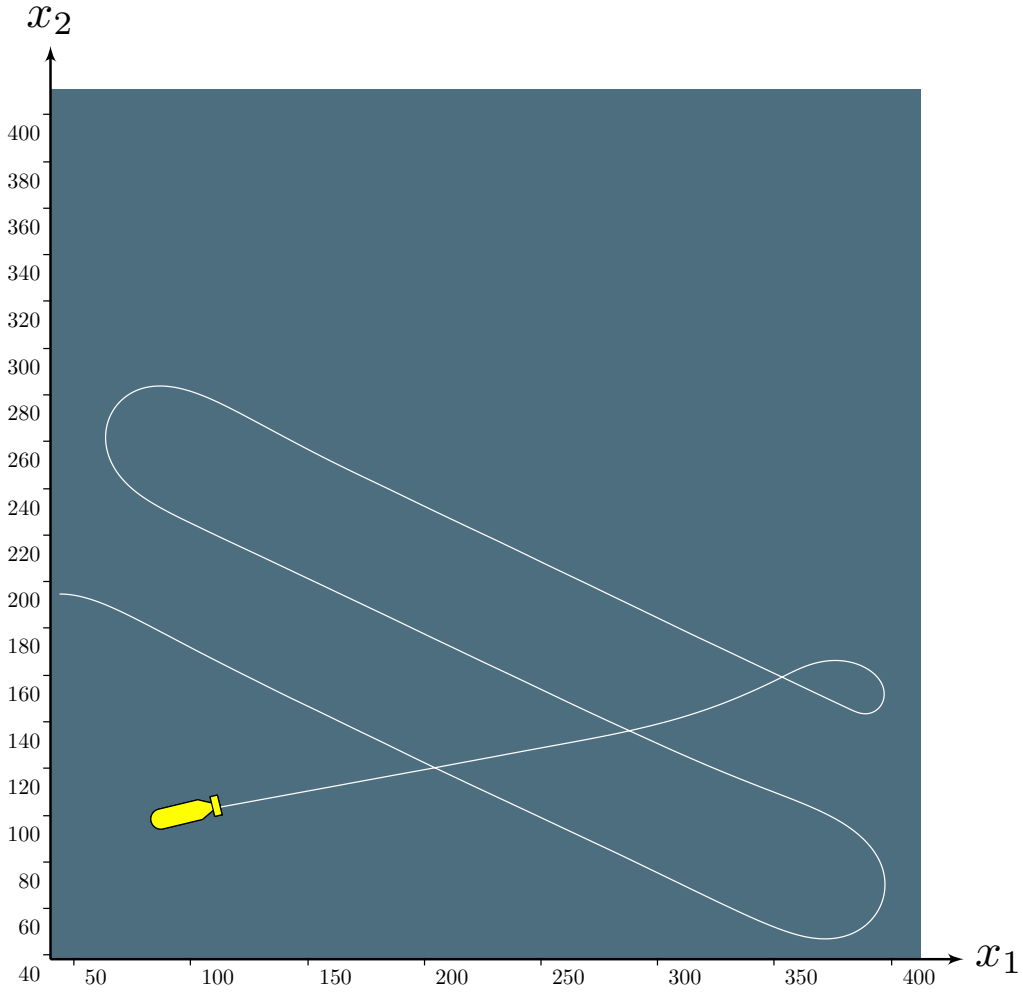


- new variable: **trajectory** $x(\cdot)$
- new domain (set): **tube** $[x](\cdot)$, interval of trajectories

Dead reckoning with actual data

Video

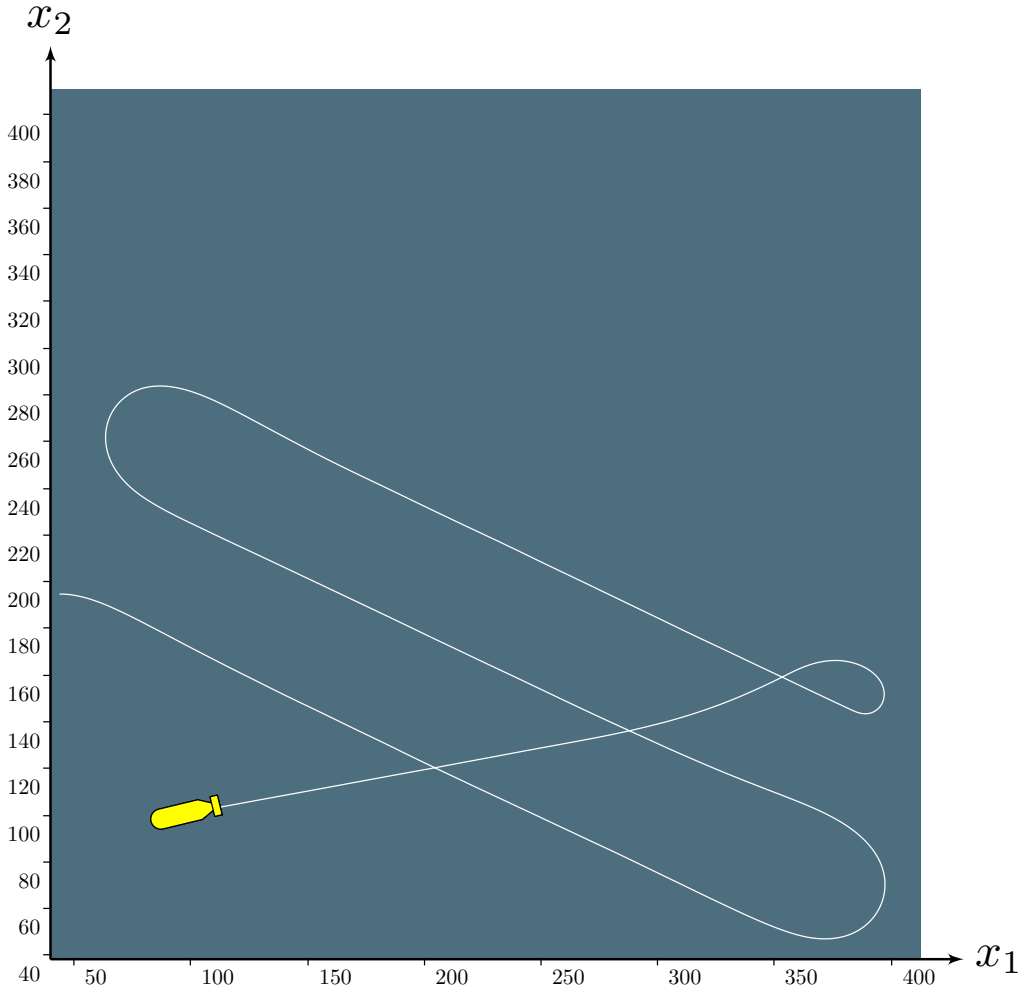
Dynamic state estimation



State estimation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) \\ \end{cases}$$

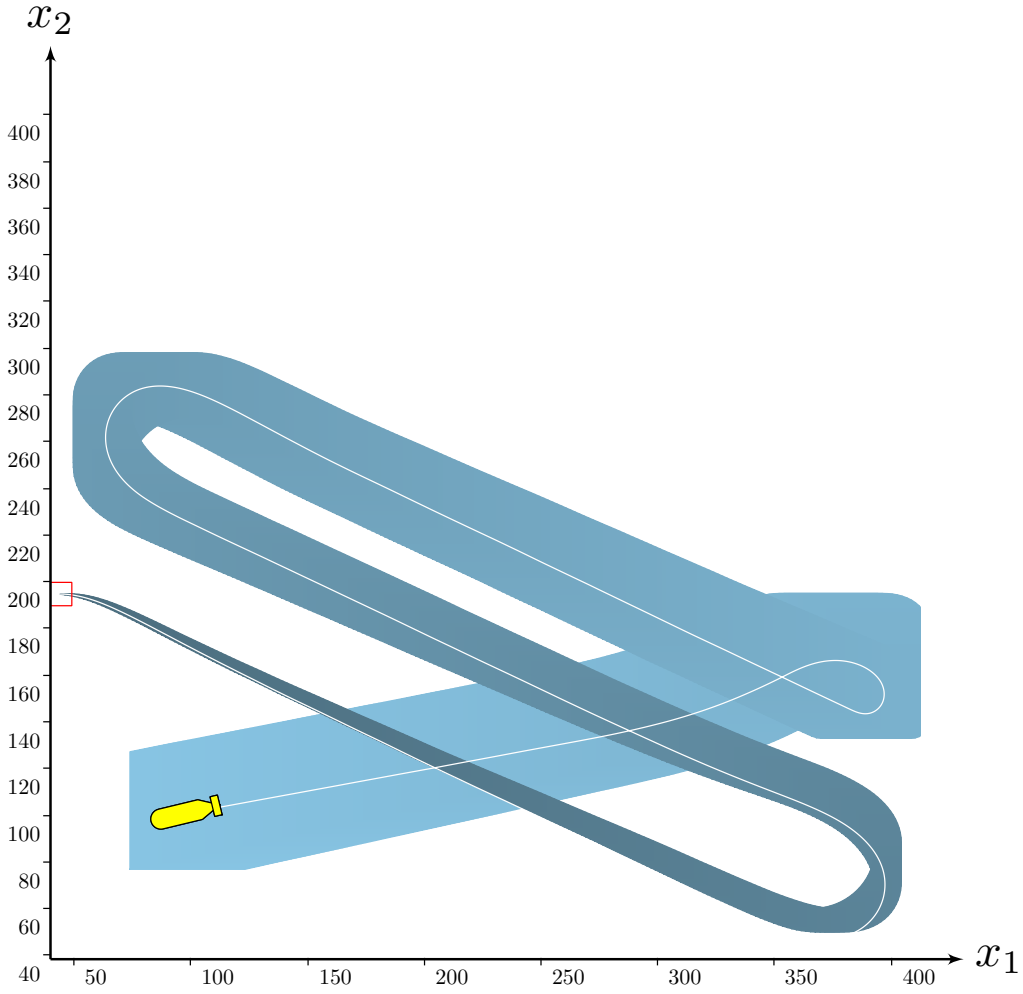
Dynamic state estimation



State estimation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) \\ \dot{\mathbf{x}}(t) = \mathbf{v}(t) \end{cases}$$

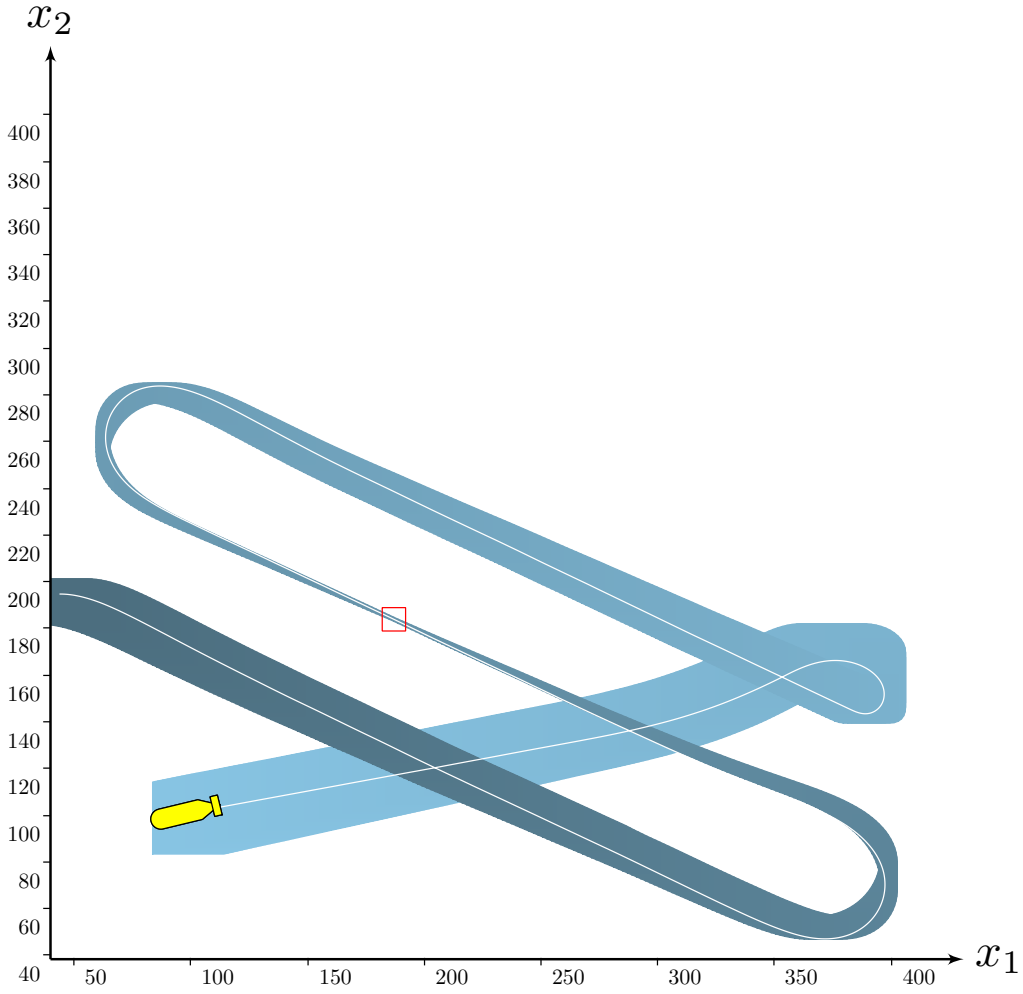
Dynamic state estimation



State estimation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) \\ \dot{\mathbf{x}}(t) = \mathbf{v}(t) \\ \mathbf{x}(t_0) \in [\mathbf{x}_0] \end{cases}$$

Dynamic state estimation



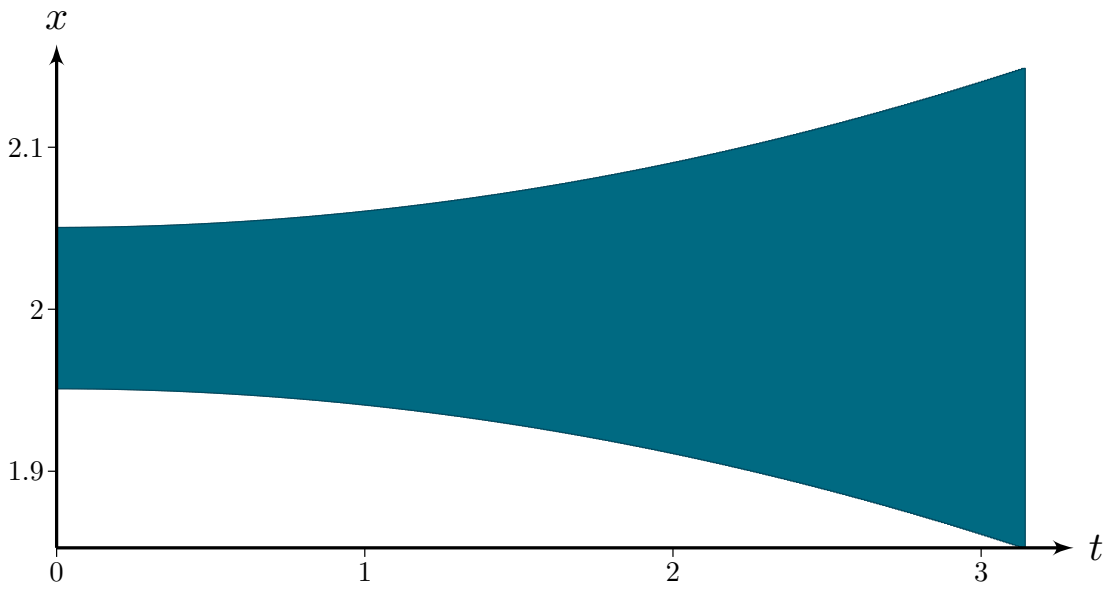
State estimation:

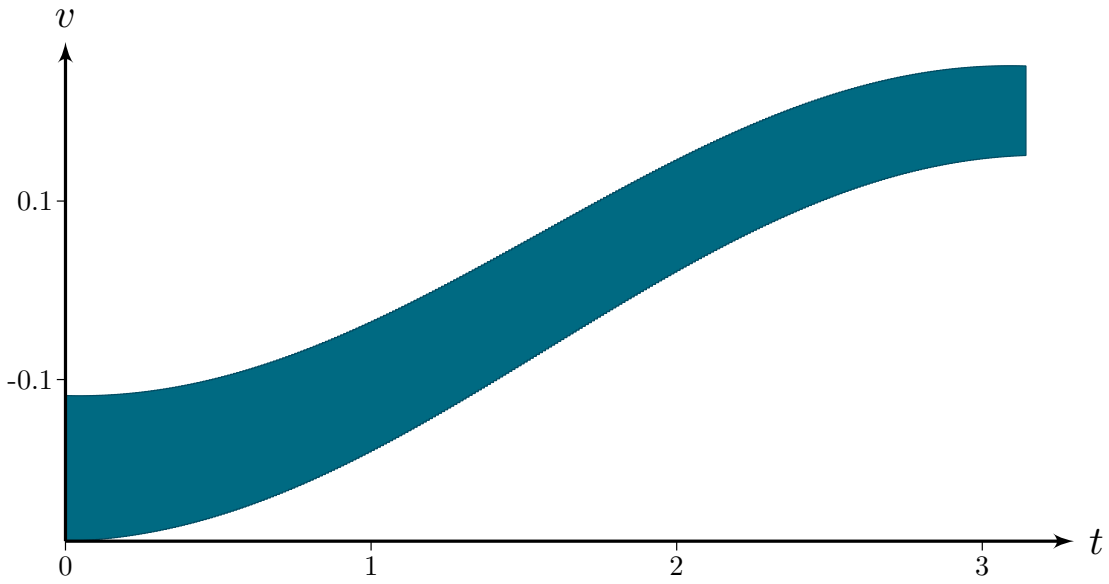
$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) \\ \dot{\mathbf{x}}(t) = \mathbf{v}(t) \\ \mathbf{x}(t_1) \in [\mathbf{x}_1] \end{cases}$$

Derivative constraint

Differential constraint:

- $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$
- one trajectory and its derivative





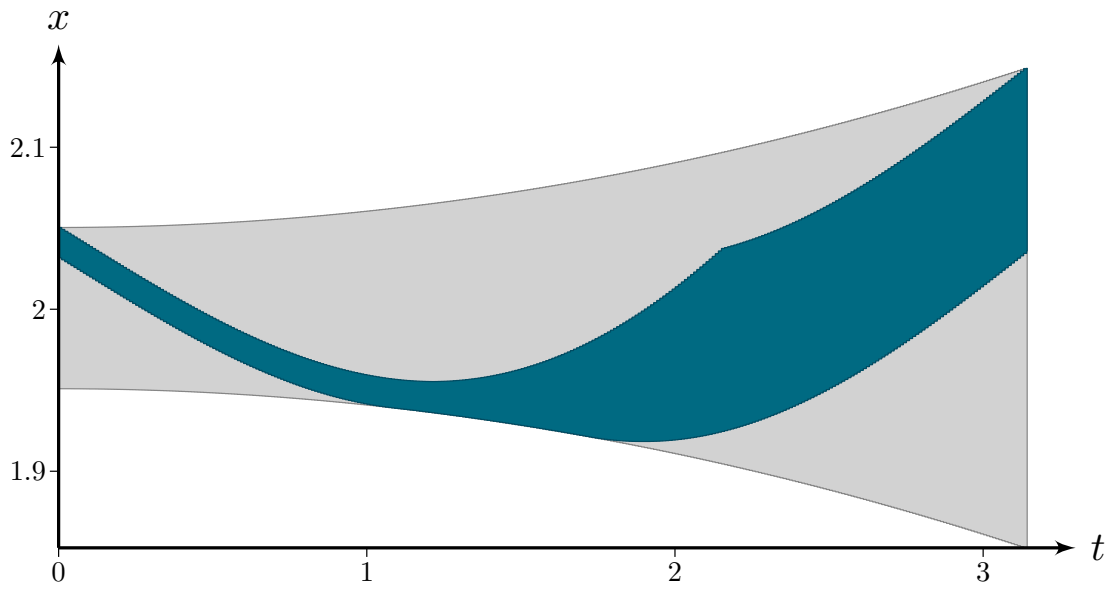
Derivative constraint

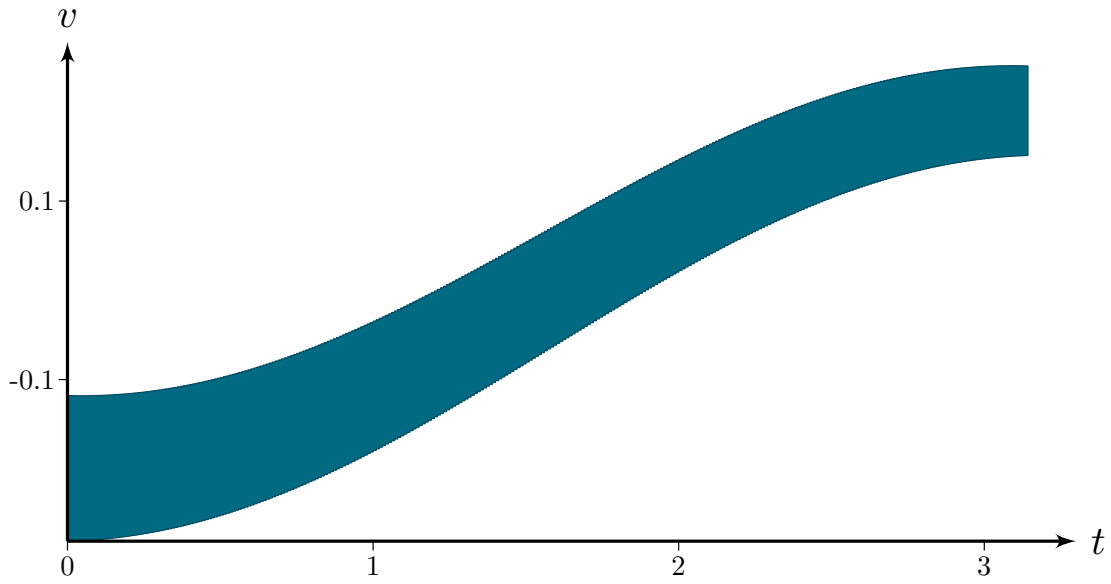
Differential constraint:

- $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$
- one trajectory and its derivative

Related contractor:

- $\mathcal{C}_{\texttt{deriv}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$





Guaranteed computation of robot trajectories

Rohou, Jaulin, Mihaylova, Le Bars, Veres

Robotics and Autonomous Systems, 2017

Trajectory evaluation constraint

Trajectory evaluation $\begin{cases} \mathbf{z} = \mathbf{y}(t) \\ \\ \end{cases}$

Reliable non-linear state estimation involving time uncertainties
Rohou, Jaulin, Mihaylova, Le Bars, Veres *Automatica*, 2018

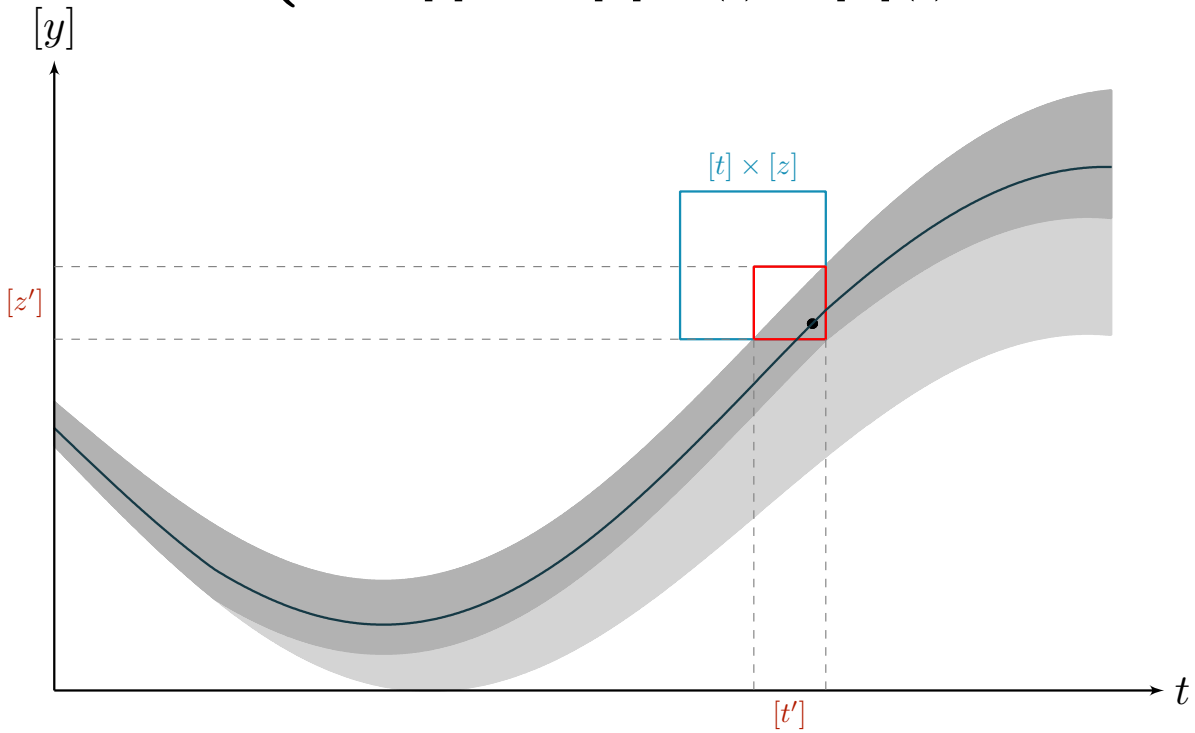
Trajectory evaluation constraint

$$\text{Trajectory evaluation} \begin{cases} \mathbf{z} = \mathbf{y}(t) \\ \\ t \in [t], \mathbf{z} \in [\mathbf{z}], \mathbf{y}(\cdot) \in [\mathbf{y}](\cdot) \end{cases}$$

Reliable non-linear state estimation involving time uncertainties
Rohou, Jaulin, Mihaylova, Le Bars, Veres *Automatica*, 2018

Trajectory evaluation constraint

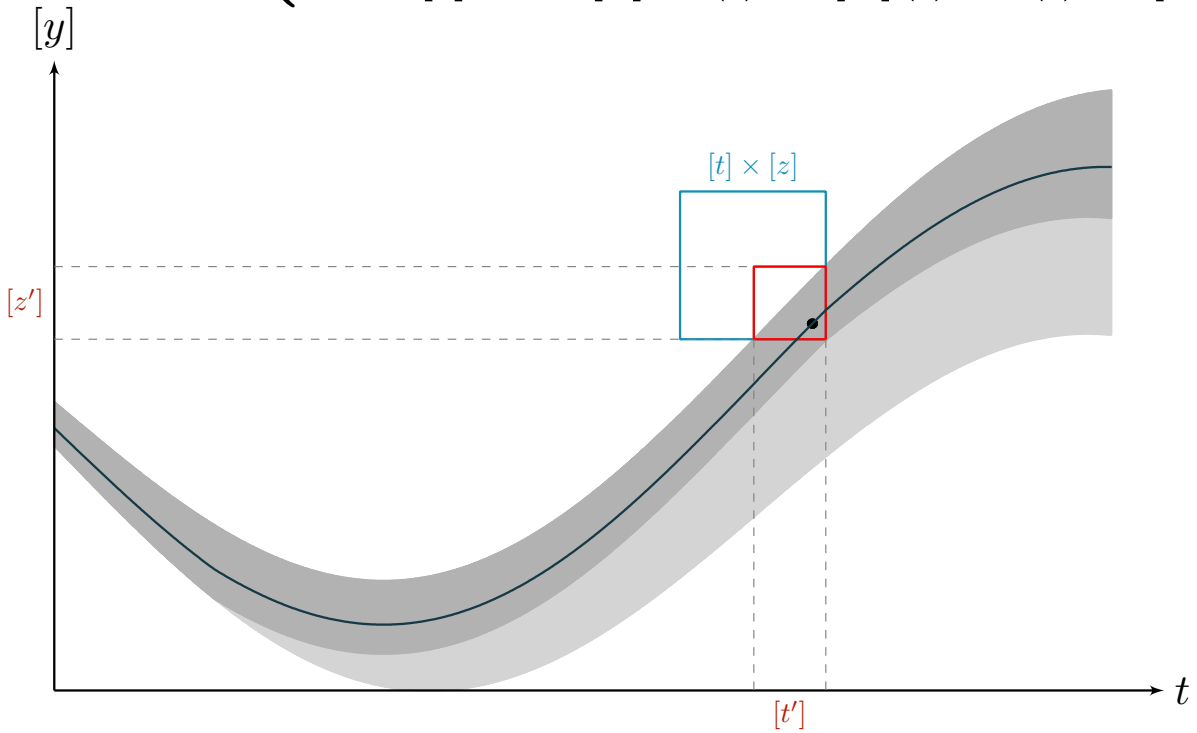
Trajectory evaluation $\begin{cases} \mathbf{z} = \mathbf{y}(t) \\ t \in [t], \mathbf{z} \in [\mathbf{z}], \mathbf{y}(\cdot) \in [\mathbf{y}](\cdot) \end{cases}$



Reliable non-linear state estimation involving time uncertainties
Rohou, Jaulin, Mihaylova, Le Bars, Veres *Automatica*, 2018

Trajectory evaluation constraint

Trajectory evaluation $\begin{cases} \mathbf{z} = \mathbf{y}(t) \\ \dot{\mathbf{y}}(\cdot) = \mathbf{w}(\cdot) \\ t \in [t], \mathbf{z} \in [\mathbf{z}], \mathbf{y}(\cdot) \in [\mathbf{y}](\cdot), \mathbf{w}(\cdot) \in [\mathbf{w}](\cdot) \end{cases}$

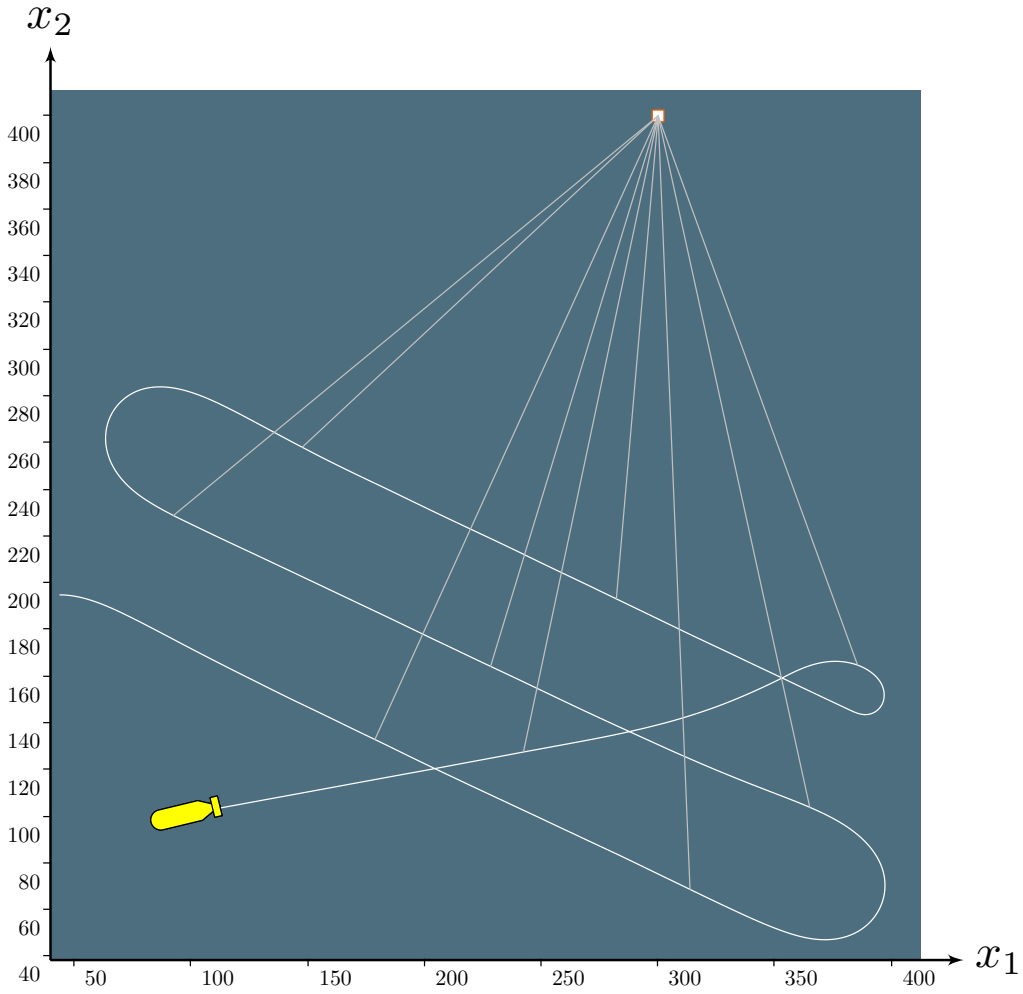


Contractor programming: $\mathcal{C}_{\text{eval}}\big([t], [\mathbf{z}], [\mathbf{y}](\cdot), [\mathbf{w}](\cdot)\big)$

Reliable non-linear state estimation involving time uncertainties
Rohou, Jaulin, Mihaylova, Le Bars, Veres *Automatica*, 2018

Dynamic state estimation

Considering **range-only measurements** from a known beacon.

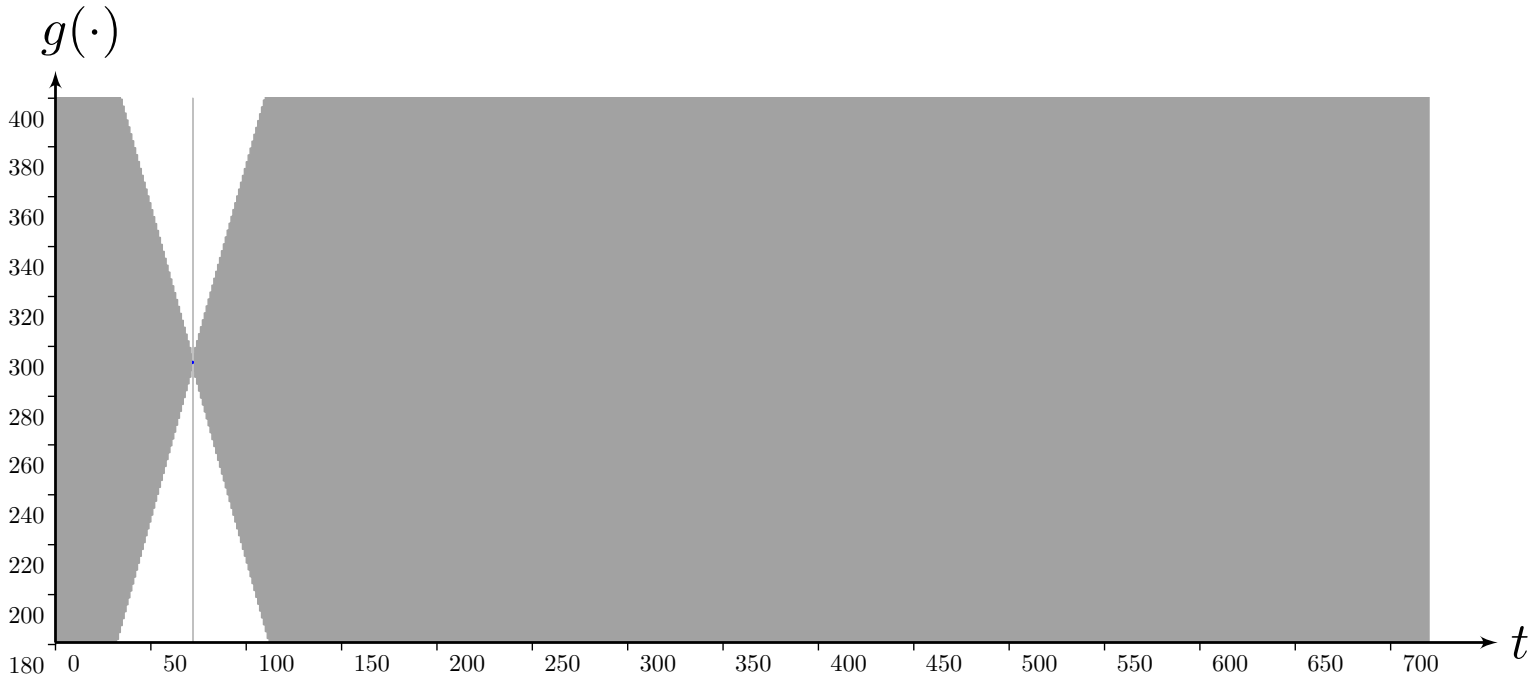


Nonlinear state estimation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) \\ y_i = g\big(\mathbf{x}(t_i), \mathbf{b}\big) \end{cases}$$

Exteroceptive measurements

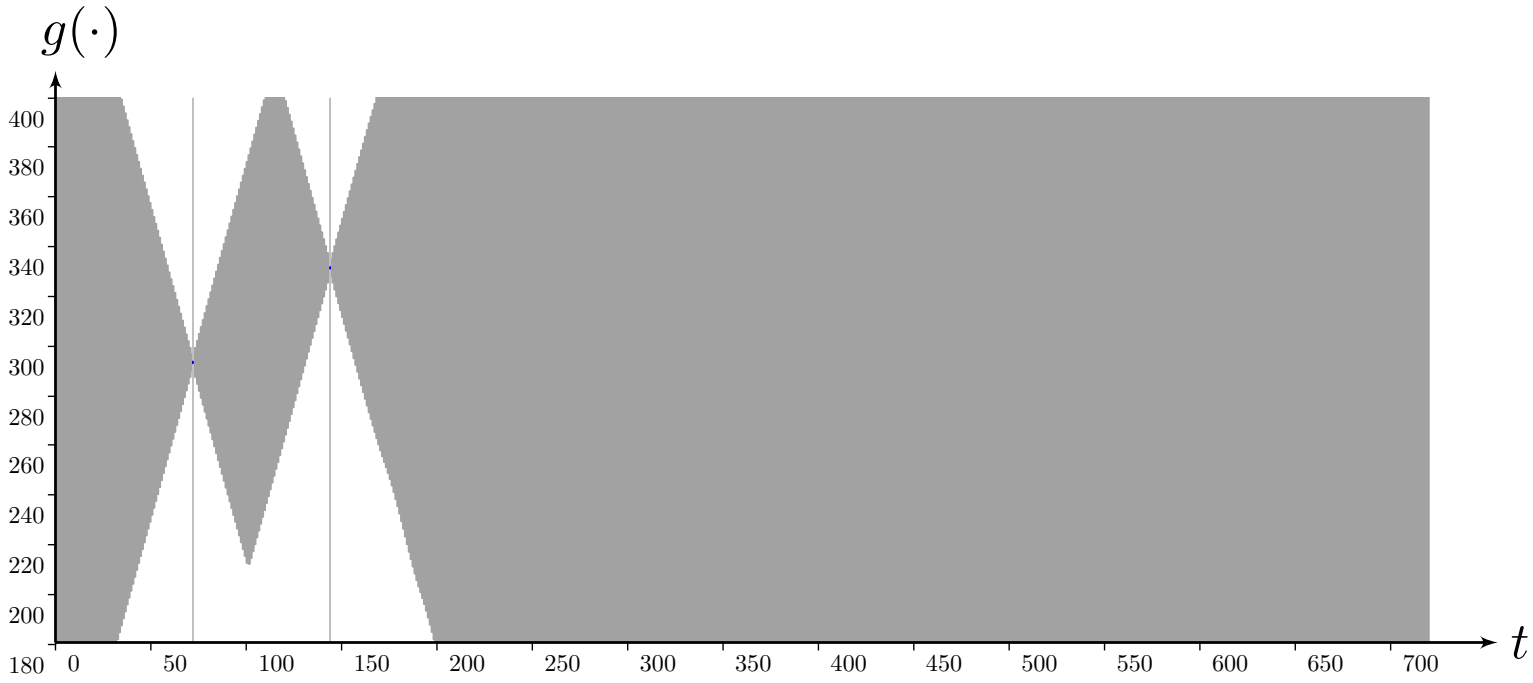
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 1 range-only measurement from the beacon.

Exteroceptive measurements

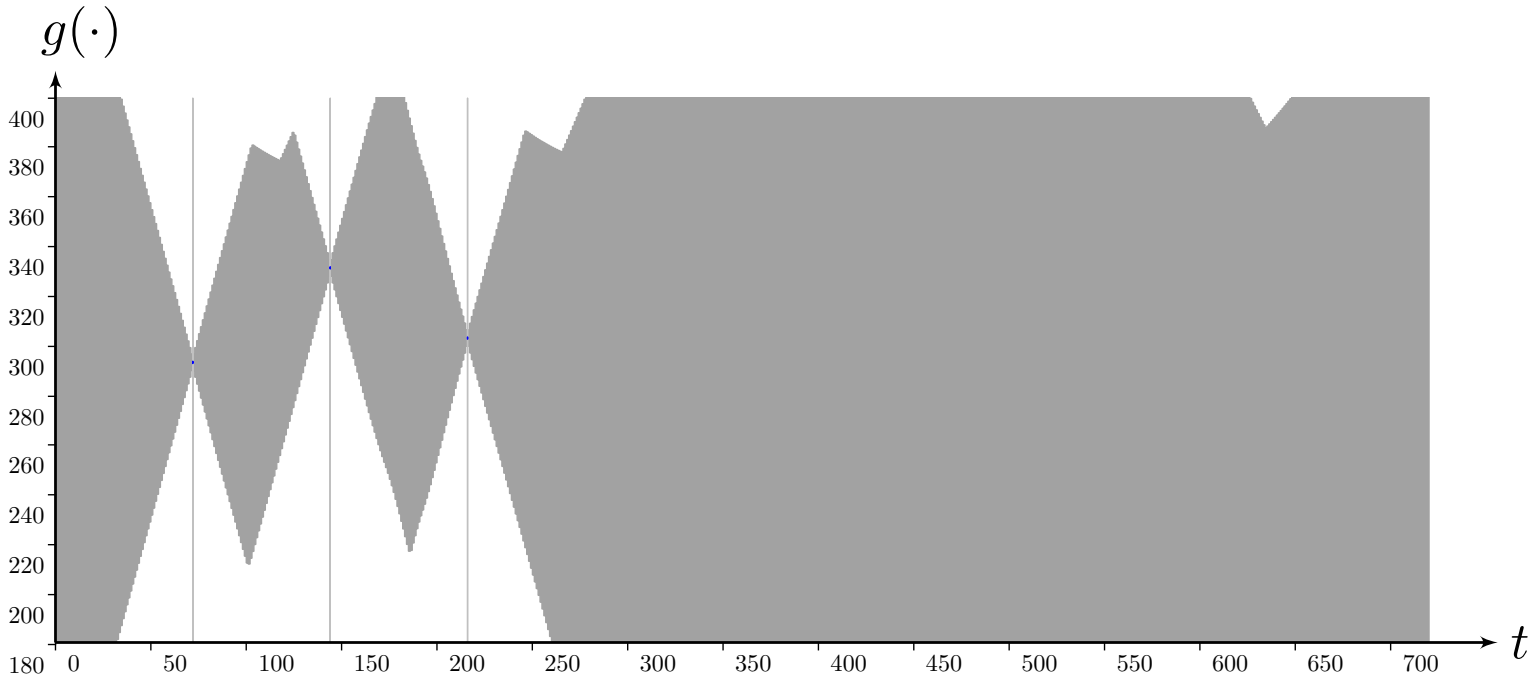
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 2 range-only measurements from the beacon.

Exteroceptive measurements

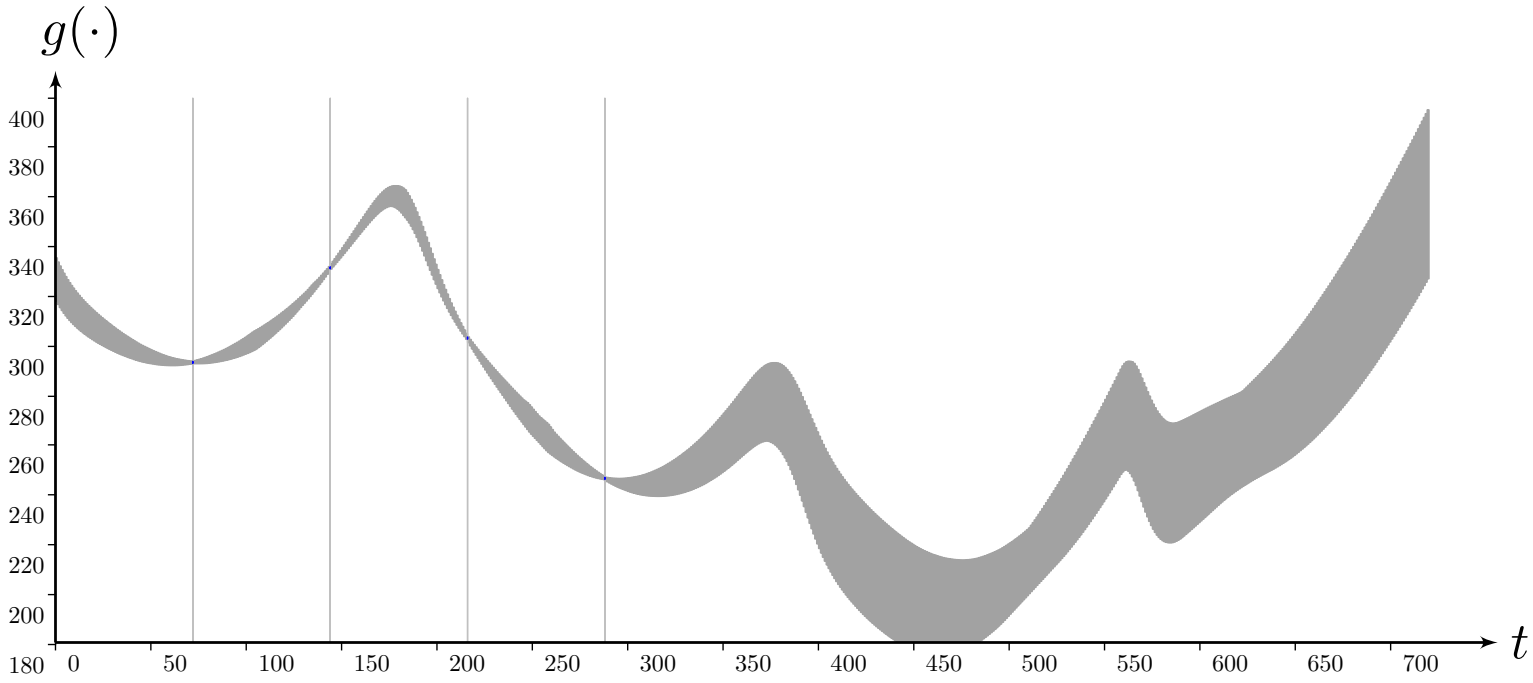
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 3 range-only measurements from the beacon.

Exteroceptive measurements

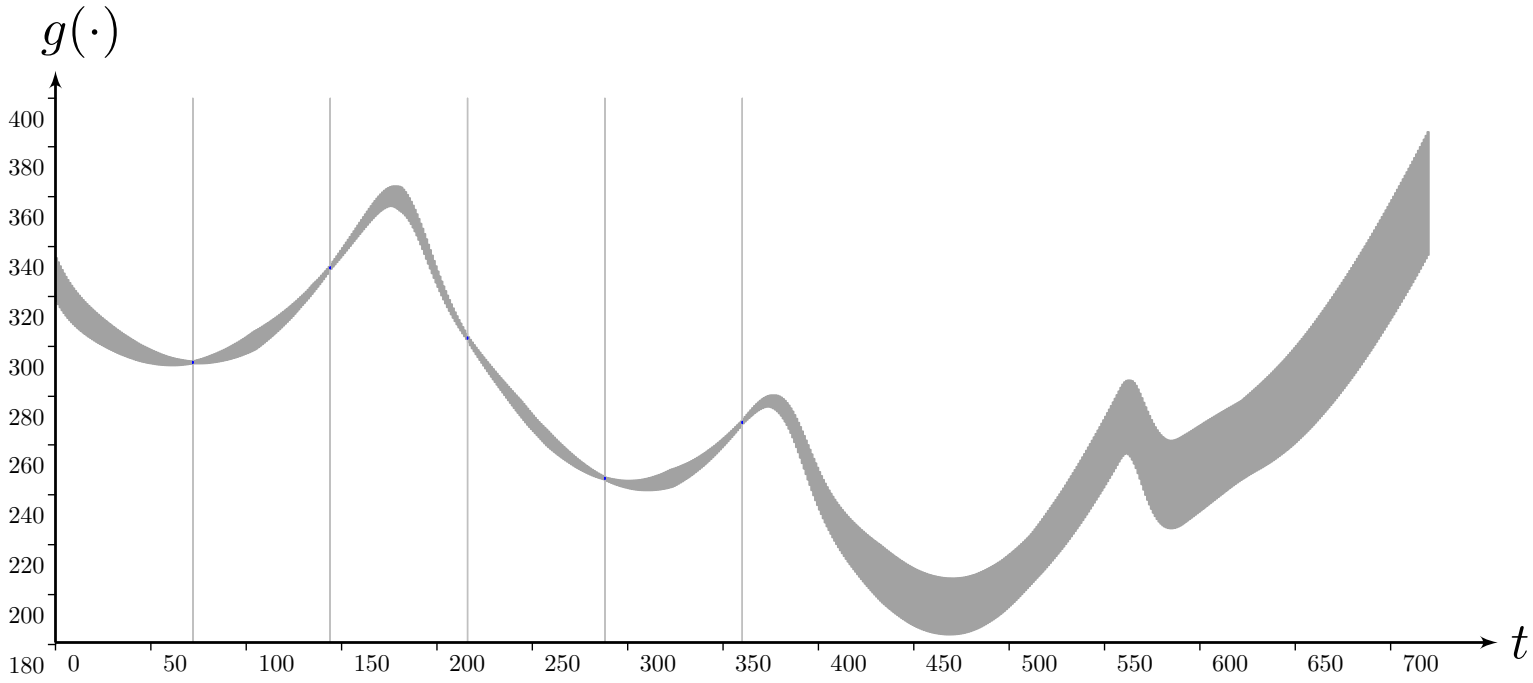
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 4 range-only measurements from the beacon.

Exteroceptive measurements

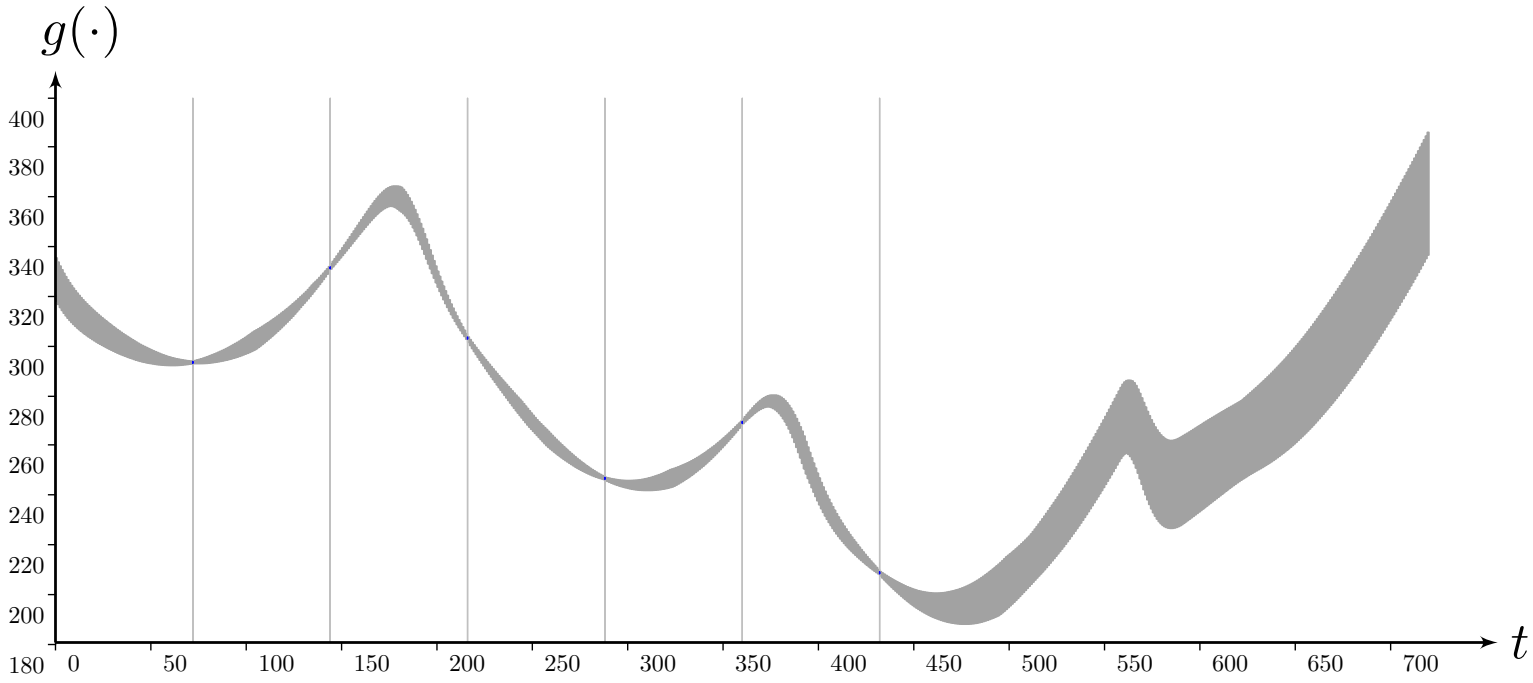
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 5 range-only measurements from the beacon.

Exteroceptive measurements

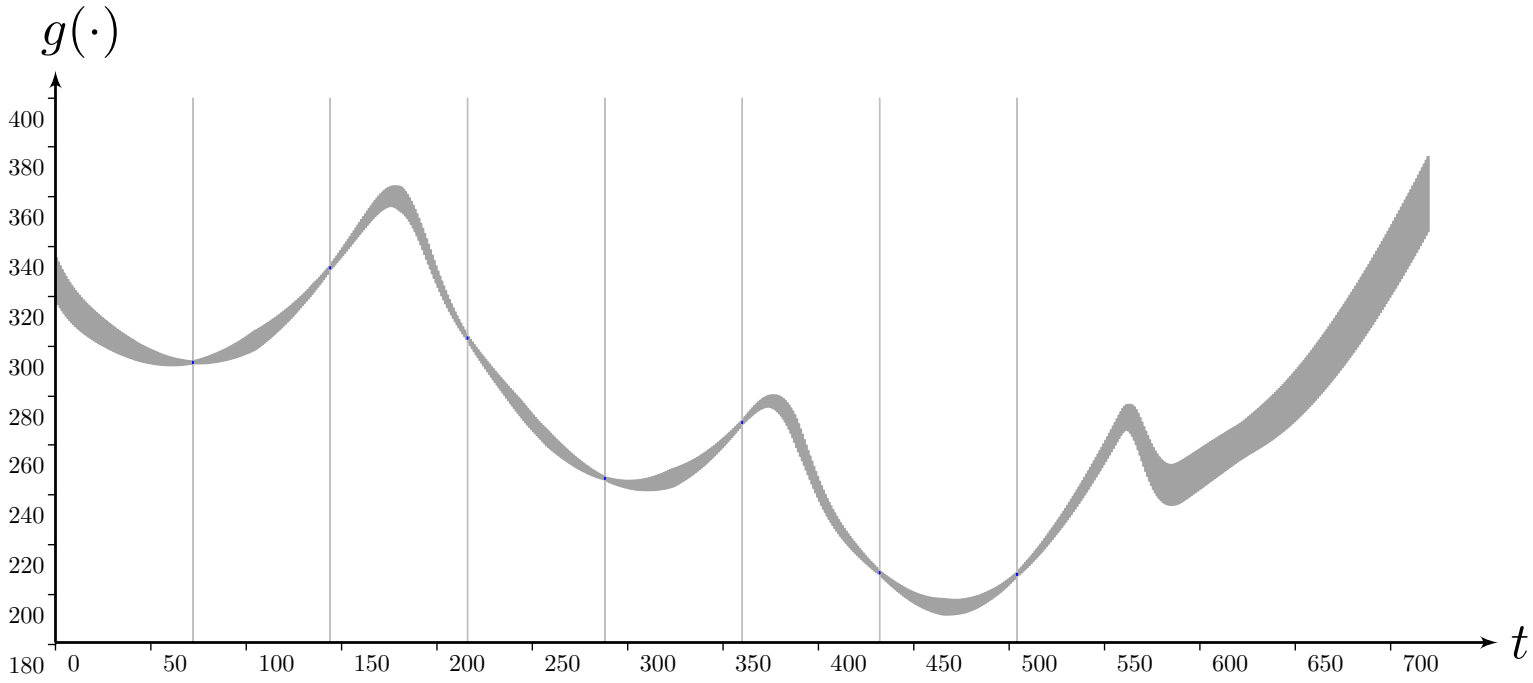
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 6 range-only measurements from the beacon.

Exteroceptive measurements

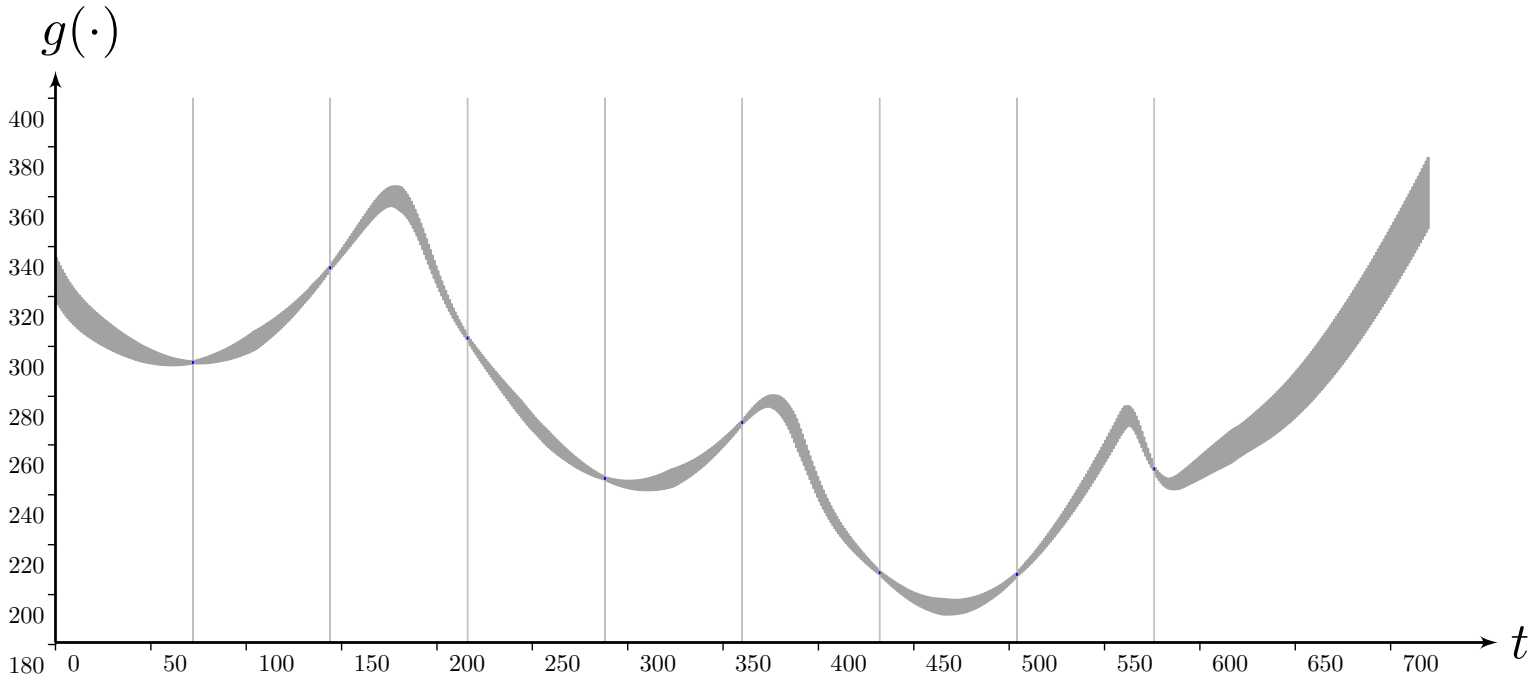
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 7 range-only measurements from the beacon.

Exteroceptive measurements

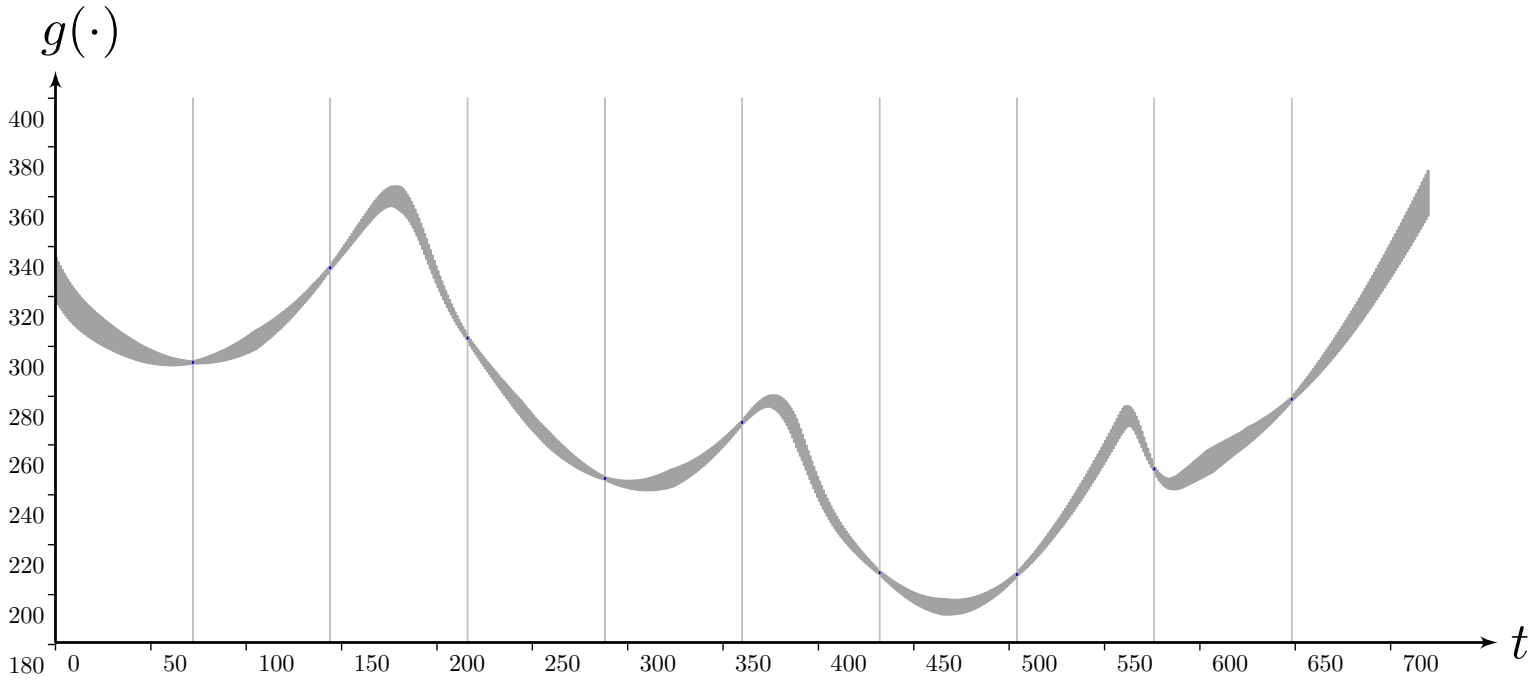
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 8 range-only measurements from the beacon.

Exteroceptive measurements

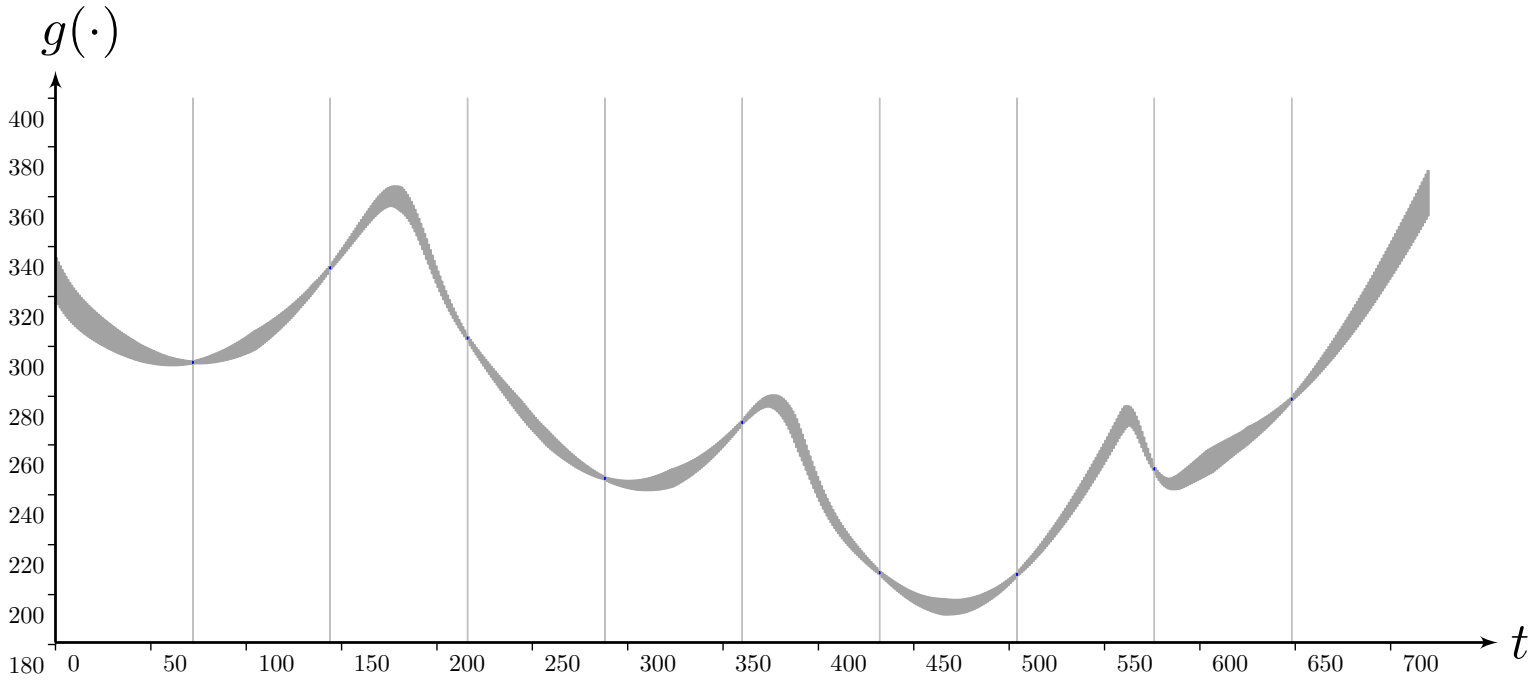
Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



Observation tube, considering 9 range-only measurements from the beacon.

Exteroceptive measurements

Creating another tube $[g](\cdot)$ that will be **constrained by measurements**.



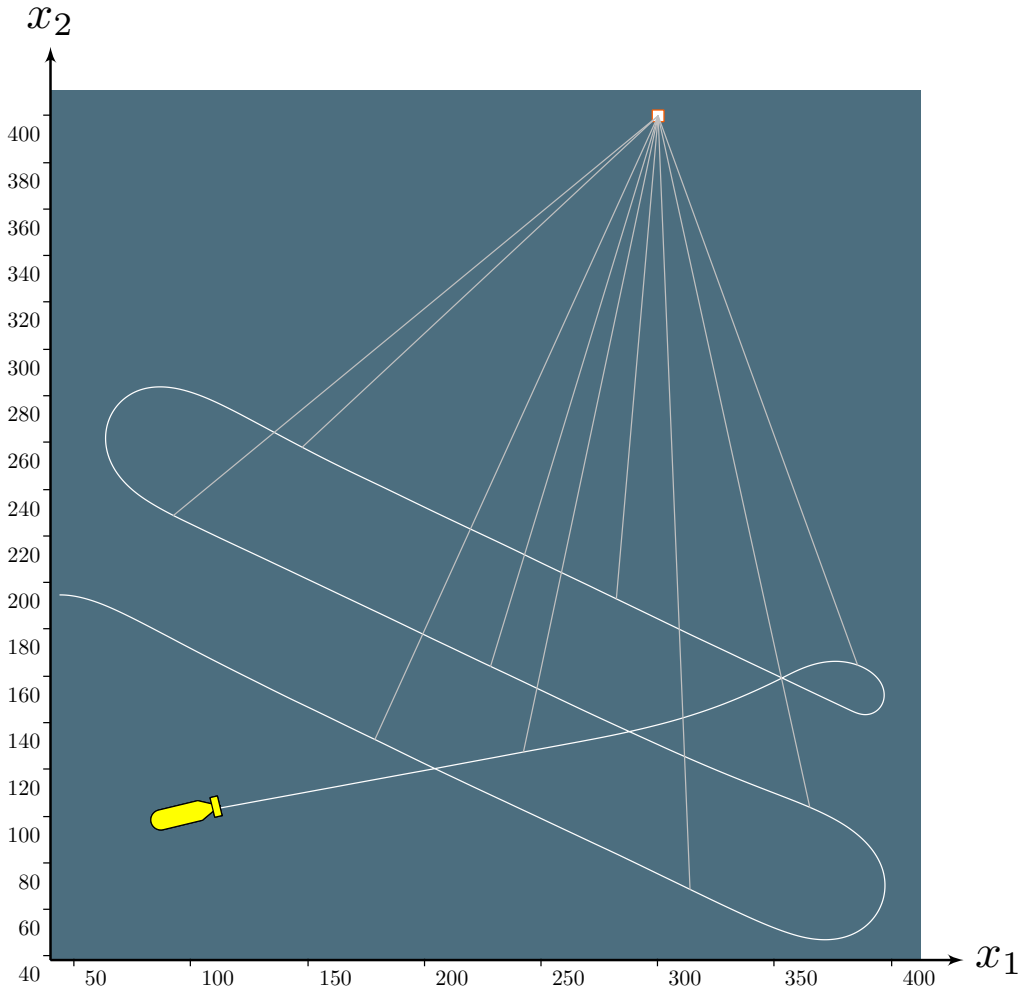
Observation tube, considering 9 range-only measurements from the beacon.

Then the state tube $[\mathbf{x}](\cdot)$ will be constrained by $[g](\cdot)$.

$$\mathcal{L}_g : \quad g(\cdot) = \sqrt{\left(x_1(\cdot) - \mathcal{B}_1\right)^2 + \left(x_2(\cdot) - \mathcal{B}_2\right)^2}.$$

Dynamic state estimation

Considering **range-only measurements** from a known beacon.



Nonlinear state estimation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) \\ y_i = g\big(\mathbf{x}(t_i), \mathbf{b}\big) \end{cases}$$

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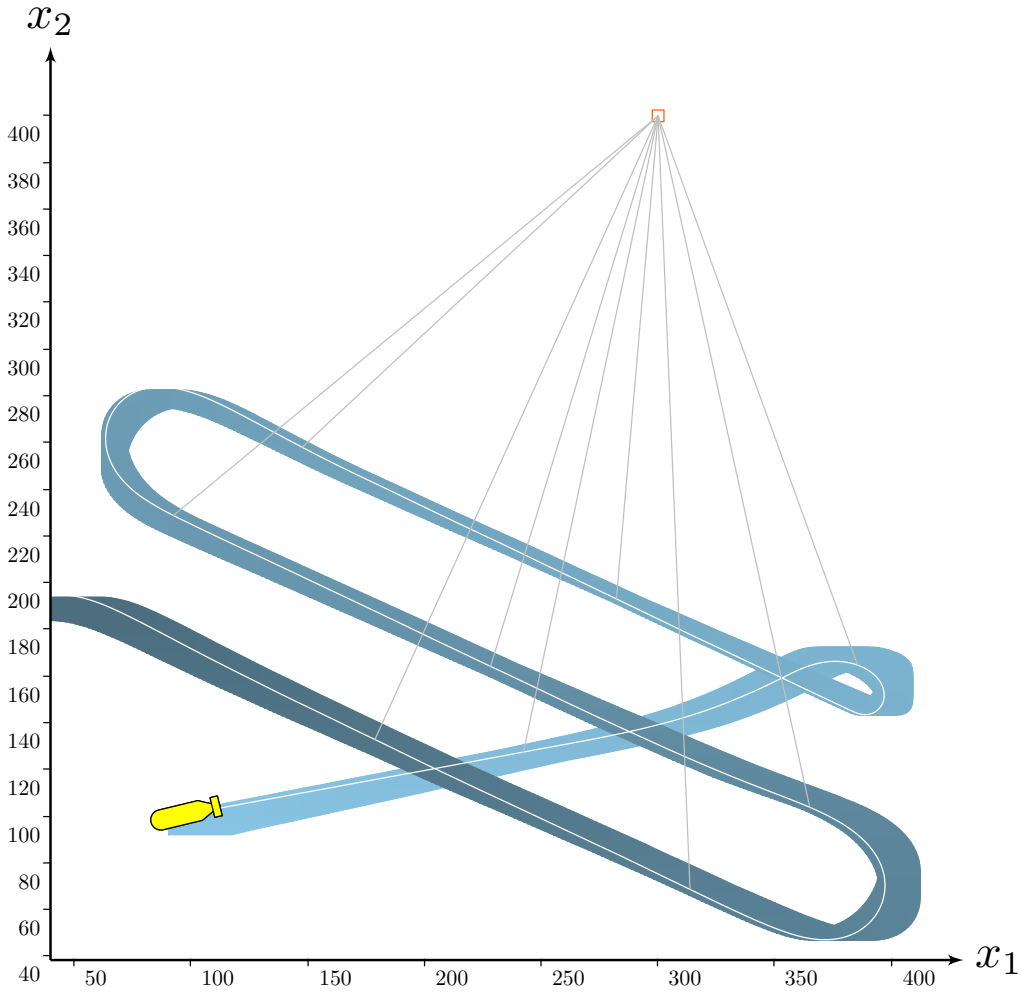

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Other example: terrain based navigation

Video

Assets of constraint programming $\times$ interval analysis

Assets of constraint programming coupled with interval analysis:

- **simplicity** of the approach

Assets of constraint programming $\times$ interval analysis

Assets of constraint programming coupled with interval analysis:

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- **reliability** of the results: no solution can be lost

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Assets of constraint programming $\times$ interval analysis

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- **complex systems** easily handled

Assets of constraint programming × interval analysis

Assets of constraint programming coupled with interval analysis:

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- focus on **the *what* instead of the *how***
- **complex systems** easily handled

Video

Section 4

The Codac library

Domains (wrappers)

- for reals $x \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$: intervals $[x]$ and boxes $[\mathbf{x}]$
- for trajectories $x(\cdot) : \mathbb{R} \to \mathbb{R}$: tubes $[x](\cdot)$

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- for subsets $\mathbb{X} \subset \mathbb{R}^n$: thicksets $\mathbb{X} \in [\mathbb{X}] = [\mathbb{X}^-, \mathbb{X}^+]$

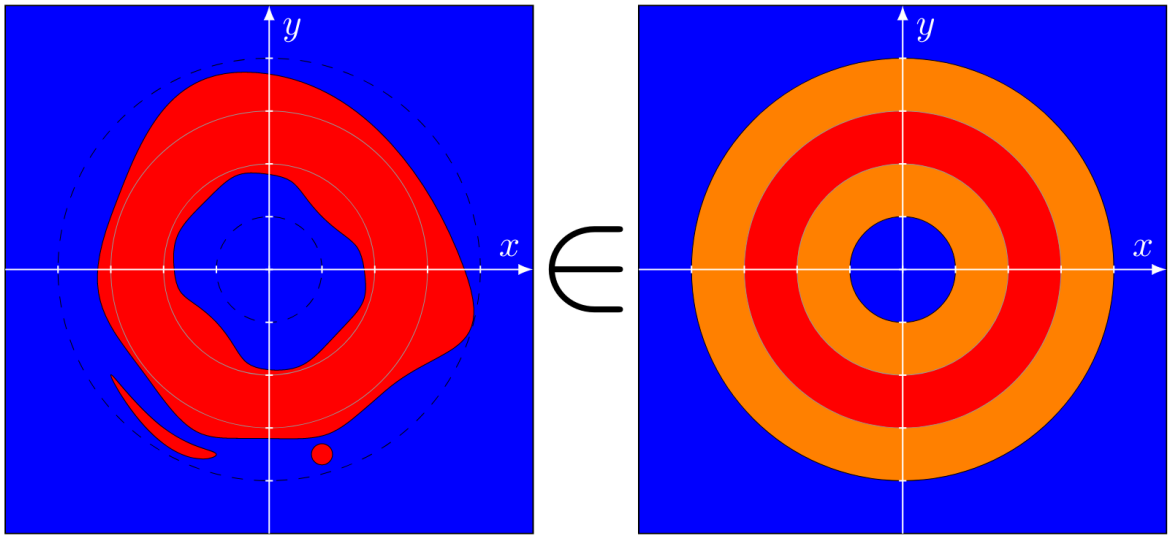

Illustration of a thickset (right-hand side)
for enclosing and uncertain red set (left-hand side)

Thick set inversion
Desrochers, Jaulin. *Artificial Intelligence. Volume 249, Issue C, Pages 1-18*, 2017

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- *etc.*

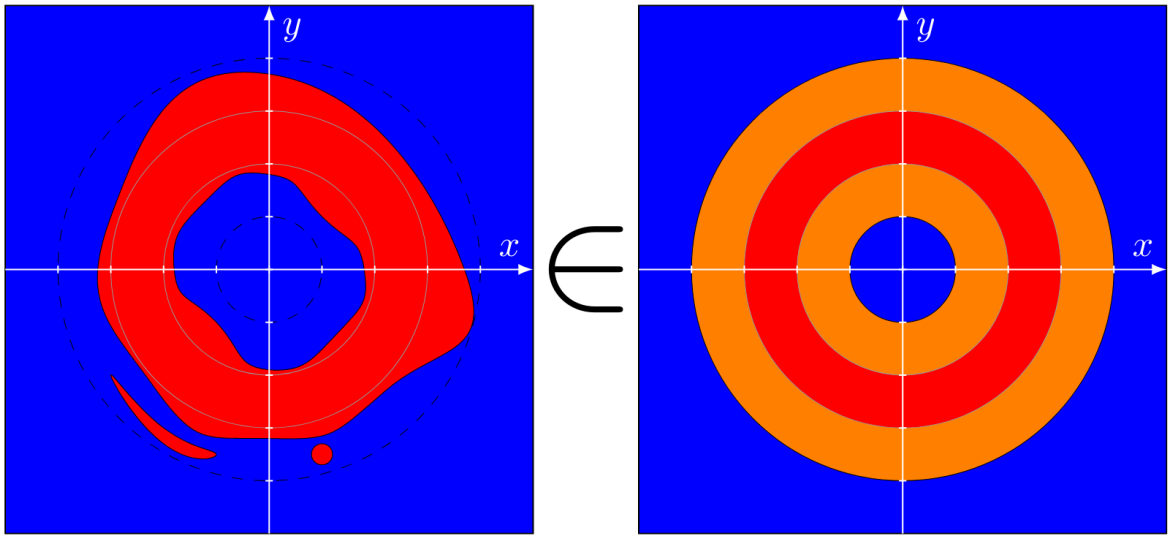

Illustration of a thickset (right-hand side)
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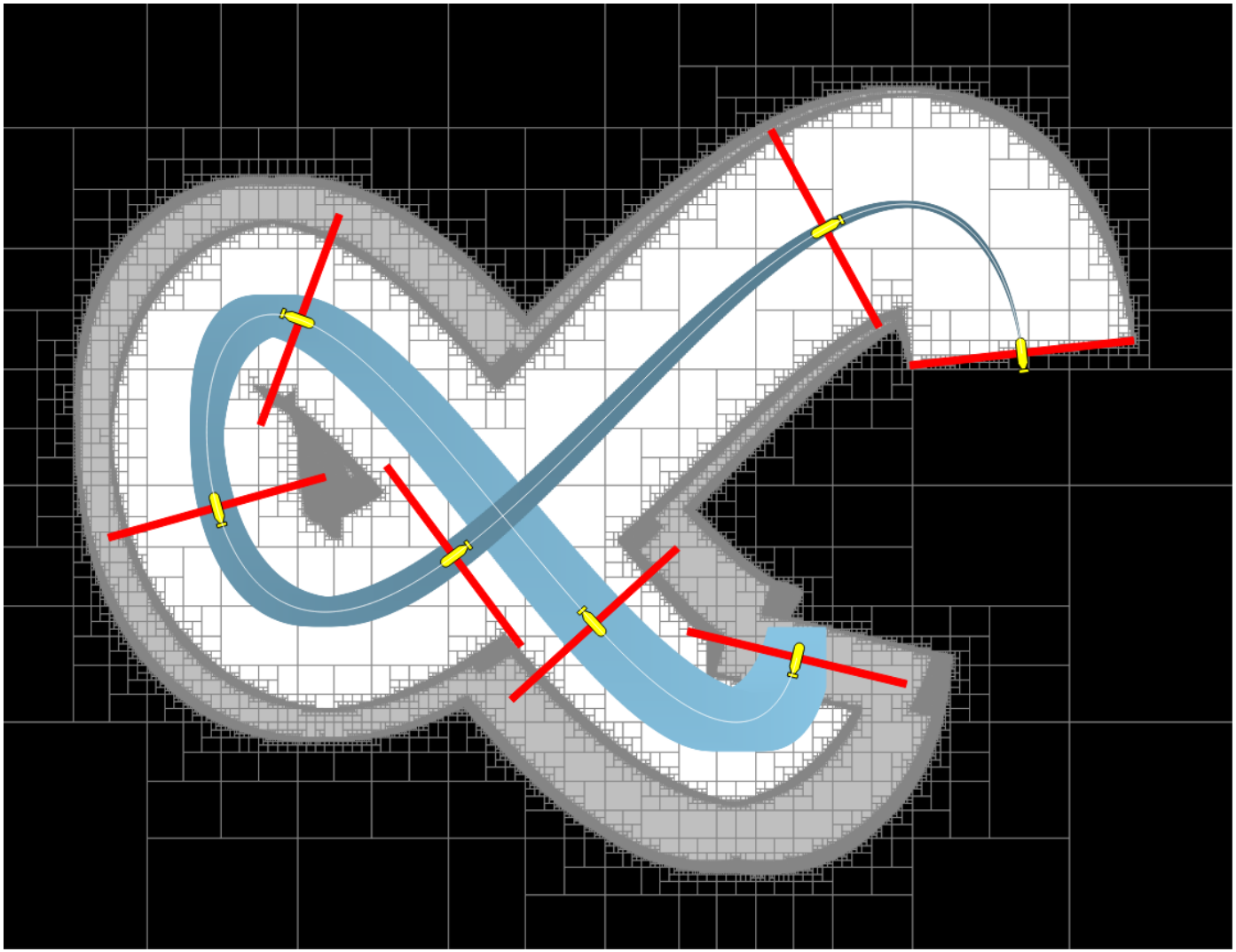
Thick set inversion
Desrochers, Jaulin. *Artificial Intelligence. Volume 249, Issue C, Pages 1-18*, 2017

Example of tubes and thicksets



Computing a Guaranteed Approximation of the Zone Explored by a Robot
Desrochers, Jaulin. *IEEE Transaction on Automatic Control. Volume 62, Issue 1, pages 425-430*, 2017

Codac: Catalog Of Domains And Contractors

Several types of **domains**:

- `Interval`, `IntervalVector`, `IntervalMatrix`
- `Tube`, `TubeVector`, `Slice`
- `Thickset`
- `Ellipsoid` (next release)
- ...

Codac: Catalog Of Domains And Contractors

Several types of **domains**:

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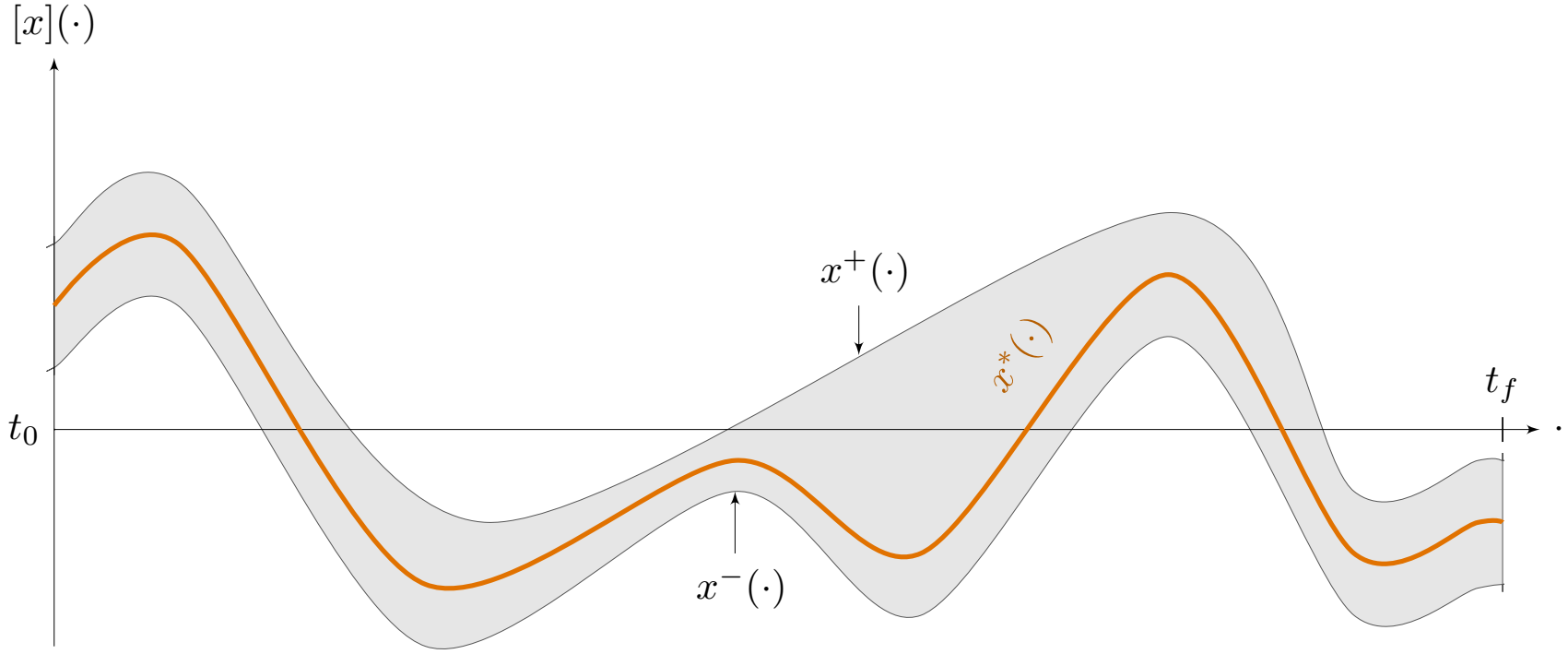
- `Ellipsoid` (next release)
- ...

Contractors for various constraints:

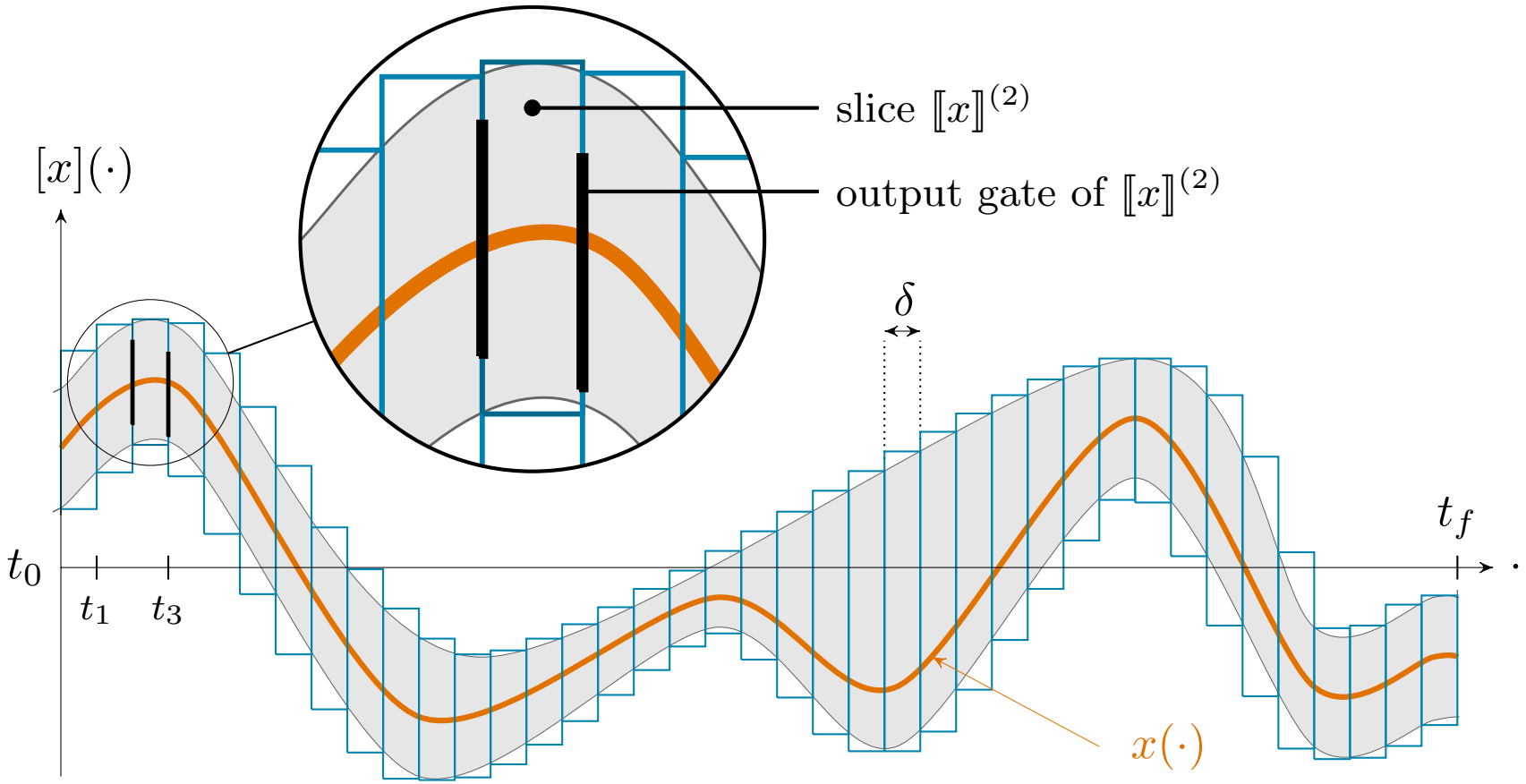
- non-linear constraints $\mathbf{f}(\mathbf{x}) = \mathbf{0}$
- geometric constraints: distance, polar equation, circles, ...
- differential equations: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$
- time uncertainties: $\mathbf{y} = \mathbf{x}(t), \quad \text{with } t \in [t]$
- delays: $x(t) = y(t - \tau)$
- ...

Domains for trajectories: tubes



Example of scalar tube: interval of two trajectories

Domains for trajectories: tubes



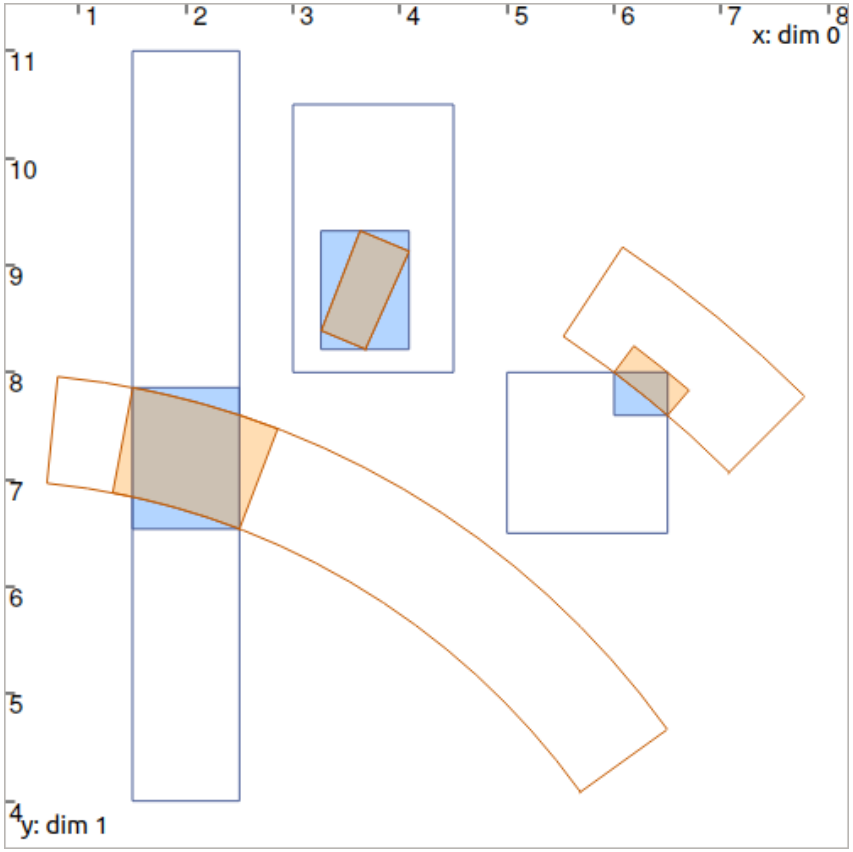
Computer implementation (http://codac.io)

Example of optimal contractors for the $\mathcal{L}_{\text{polar}}$ constraint

$$\mathcal{L}_{\text{polar}} : \begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases}$$

Optimally dealt with by:

$$\mathcal{C}_{\text{polar}}\big([x],[y],[\rho],[\theta]\big)$$



Using Codac:

```
x = Interval(..)
y = Interval(..)
r = Interval(..)
theta = Interval(..)

ctc.polar.contract(x,y,r,theta)
```

A Minimal contractor for the Polar equation: application to robot localization
Desrochers, Jaulin. *Engineering Applications of Artificial Intelligence, 55(Supplement C):83–92*, 2016

The library is open source and available:

- in Python and C++ (and now Matlab)
- on Linux, Windows, MacOS systems
- from official packages:
  Python package: `pip install codac`
  Debian in progress..: `sudo apt install libcodac`

http://www.codac.io

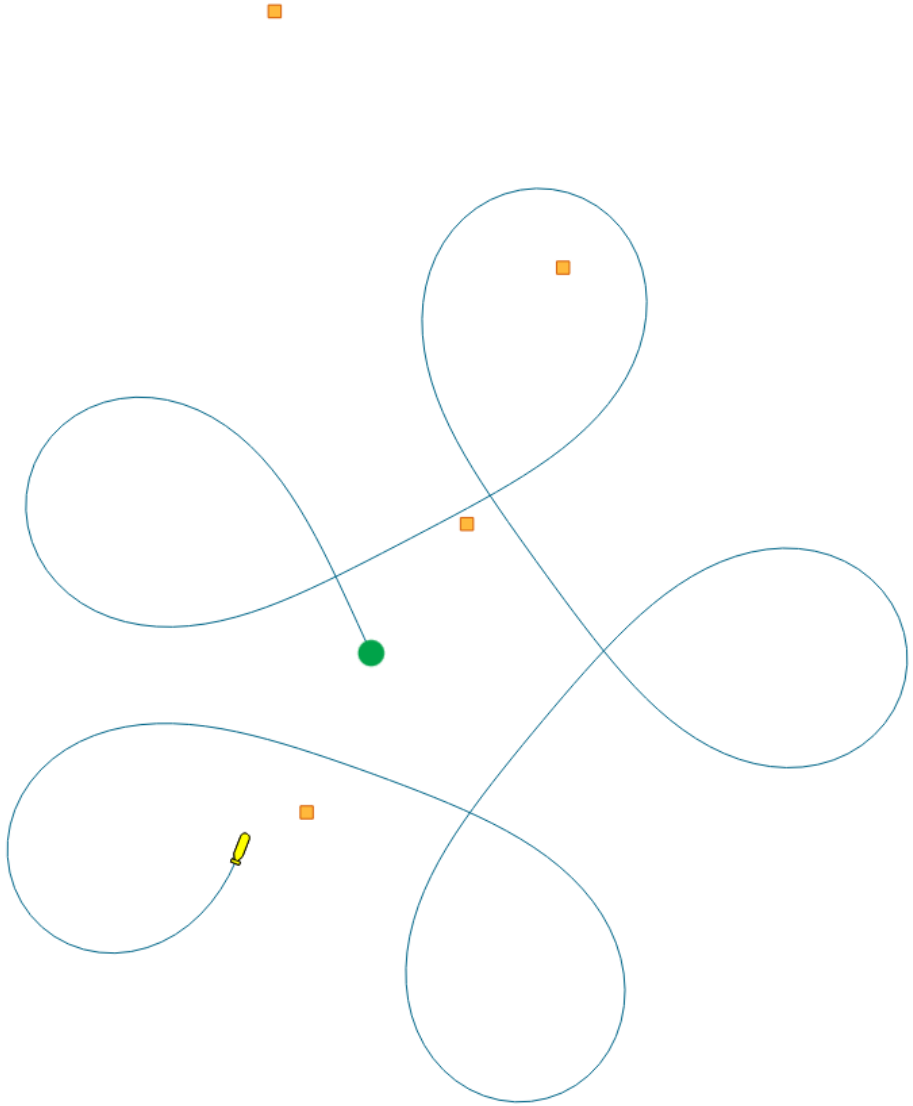
Current list of contributors

- Benoît Desrochers
- Luc Jaulin
- Gilles Chabert
- Auguste Bourgois
- Julien Damers
- Thomas Le Mézo
- Raphael Voges
- Cyril Bouvier
- Andreas Rauh
- Fabrice Le Bars
- Quentin Brateau
- Damien Massé
- Bertrand Neveu
- Peter Franek
- Gilles Trombettoni
- Aaronkumar Ehambram
- Verlein Radwan
- Mohamed Saad Ibn Seddik

Section 5

Application: range-only SLAM

Simultaneous Localization And Mapping



Formalization

SLAM: Simultaneous Localization And Mapping.
Classically, we have:

$$\begin{cases} \mathbf{x}(0) = \mathbf{0} & \text{(initial state)} \\ \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big) & \text{(evolution)} \end{cases}$$

With:

- $\mathbf{x}$: state vector (position, heading, ...)
- $\mathbf{u}$: input vector (command)
- $\mathbf{f}$: *evolution* function

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With:

- $\mathbf{x}$: state vector (position, heading, ...)
- $\mathbf{u}$: input vector (command)
- $\mathbf{f}$: *evolution* function
- g: *observation* function (scalar, distance equation)
- y_i: scalar measurements (at t_i) (distance values)
- $\mathbf{b}_j$: unknown position of a landmark

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Involved variables and domains

Variables:

- reals: $y_i \in \mathbb{R}$
- vectors: $\mathbf{b}_j \in \mathbb{R}^2$
- trajectories: $\mathbf{x}(\cdot) : \mathbb{R} \to \mathbb{R}^n$

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Domains (envelopes) of the variables:

- intervals: $[y_i] \in \mathbb{IR}$
- boxes: $[\mathbf{b}_j] \in \mathbb{IR}^2$
- tubes: $[\mathbf{x}](\cdot) : \mathbb{R} \to \mathbb{IR}^n$

Decomposition of the problem

System:

$$\begin{cases} \dot{\mathbf{x}}(\cdot) = \mathbf{f}\big(\mathbf{x}(\cdot)\big) \\ y_i = g\big(\mathbf{x}_{1,2}(t_i), \mathbf{b}_j\big) \end{cases}$$

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Elementary constraints:

- $\mathbf{v}(\cdot) = \mathbf{f}\big(\mathbf{x}(\cdot)\big) \rightarrow$ algebraic constraint $\rightarrow \mathcal{L}_{\mathbf{f}}\big(\mathbf{x}(\cdot), \mathbf{v}(\cdot)\big)$

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$\mathbf{v}(\cdot)$ and $\mathbf{p}_i$ are intermediate variables

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Note: some symbolic solver could break down such problem automatically.

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Graph of involved contractors and domains

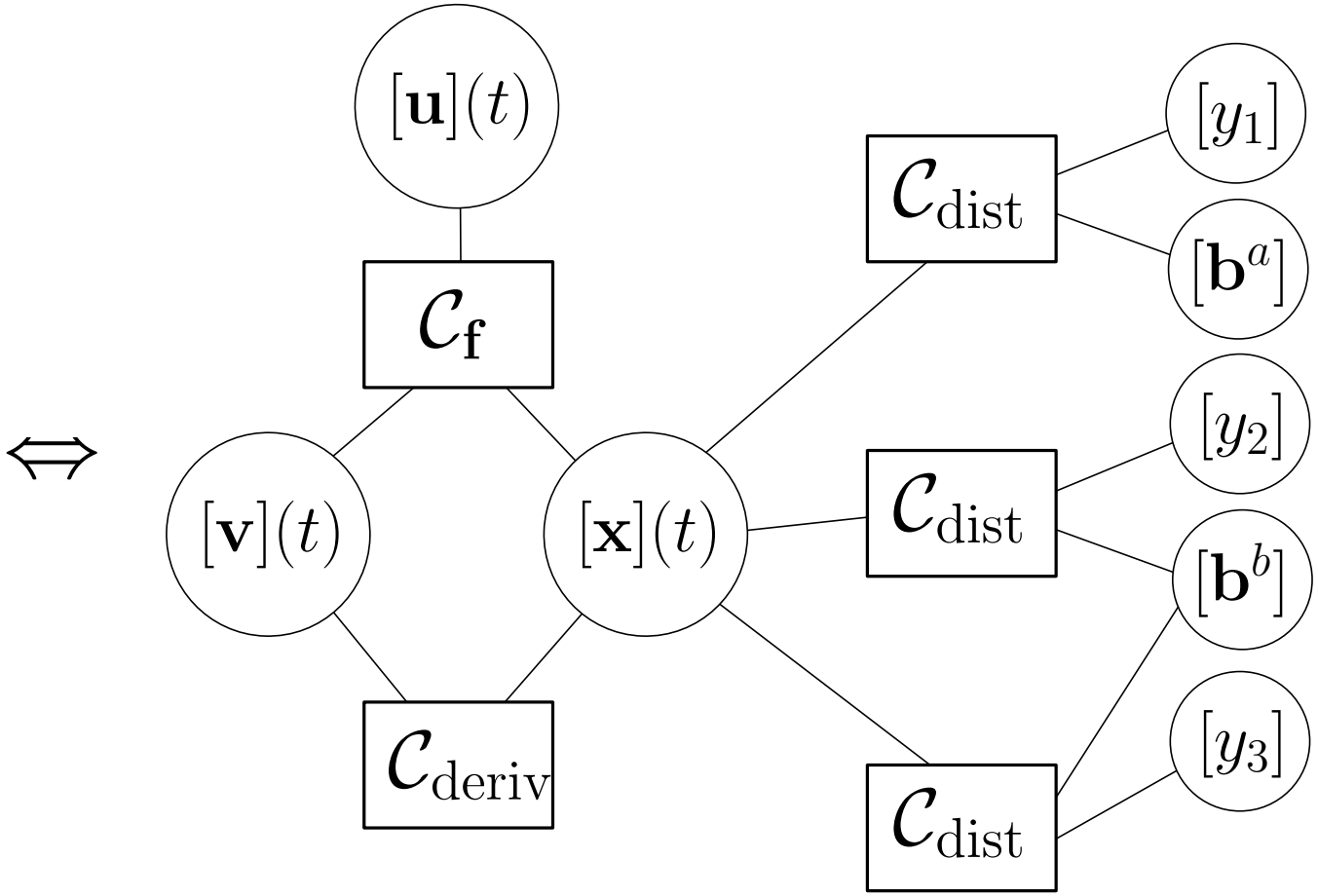
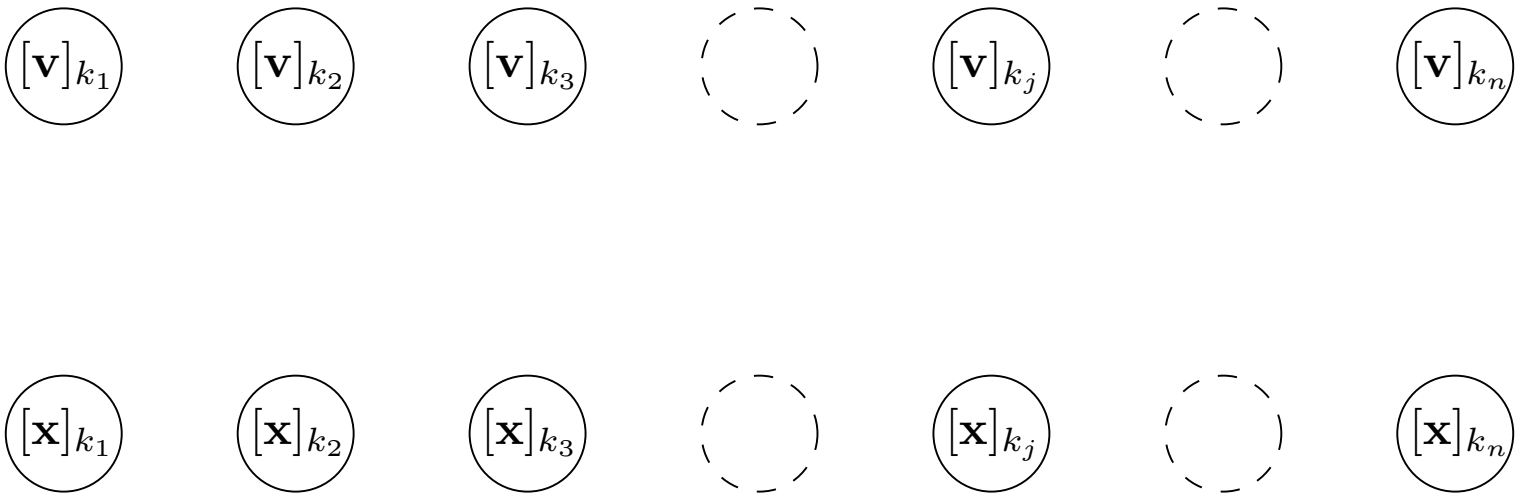


Illustration of the graph of contractors and domains corresponding to the SLAM problem: so-called **Contractor Network**.

Graph of involved contractors and domains



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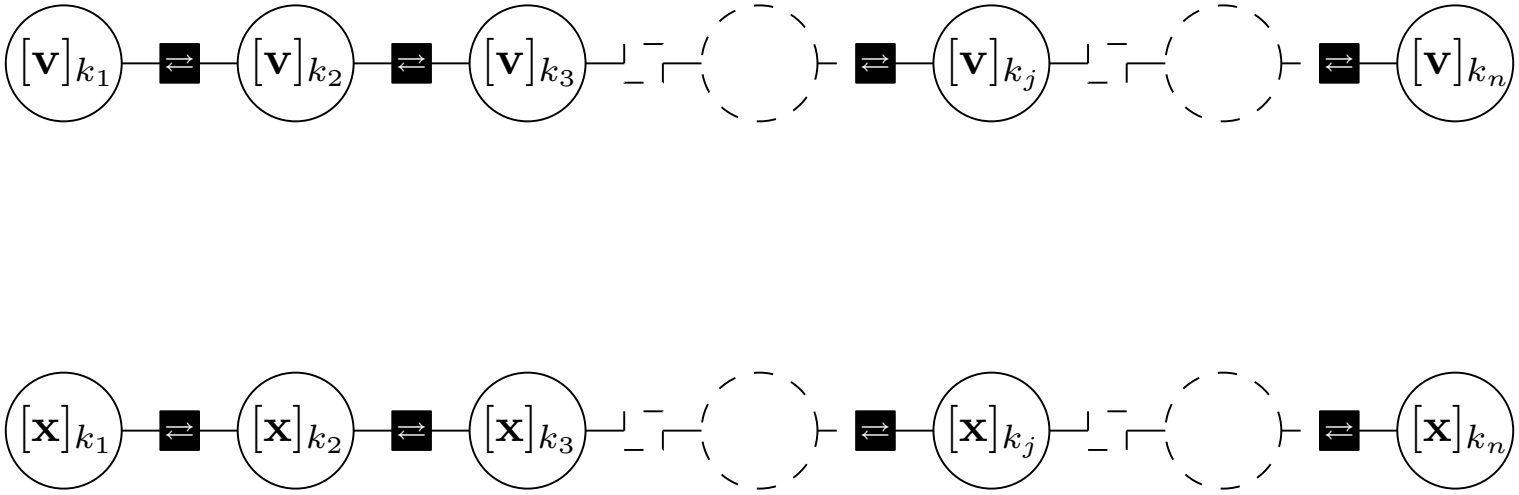

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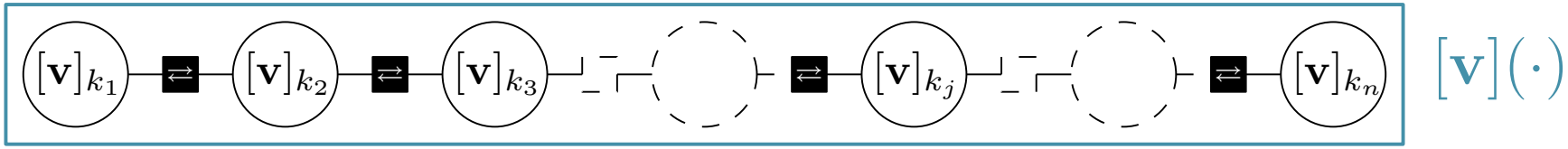

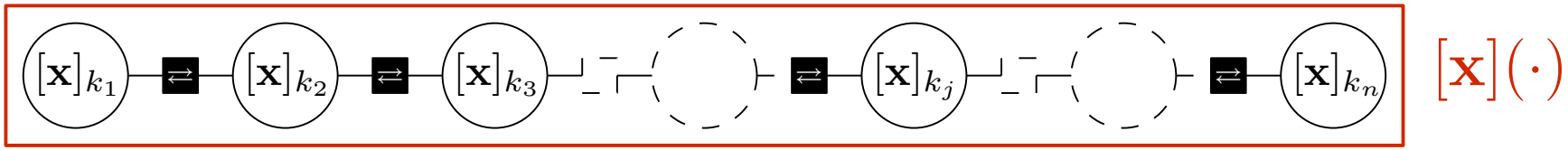

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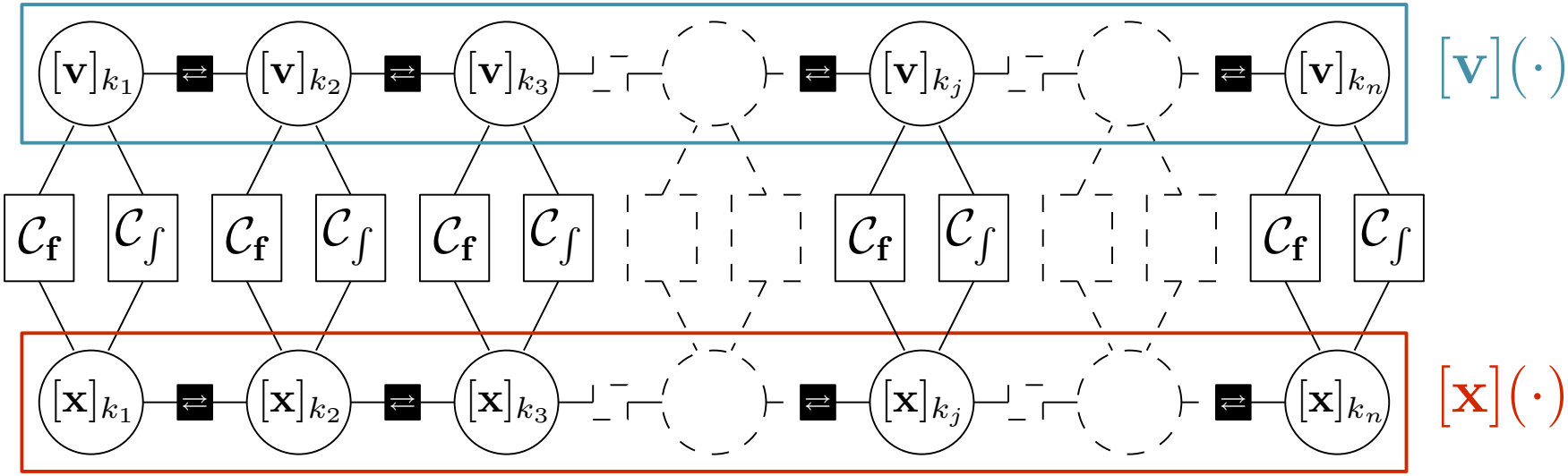

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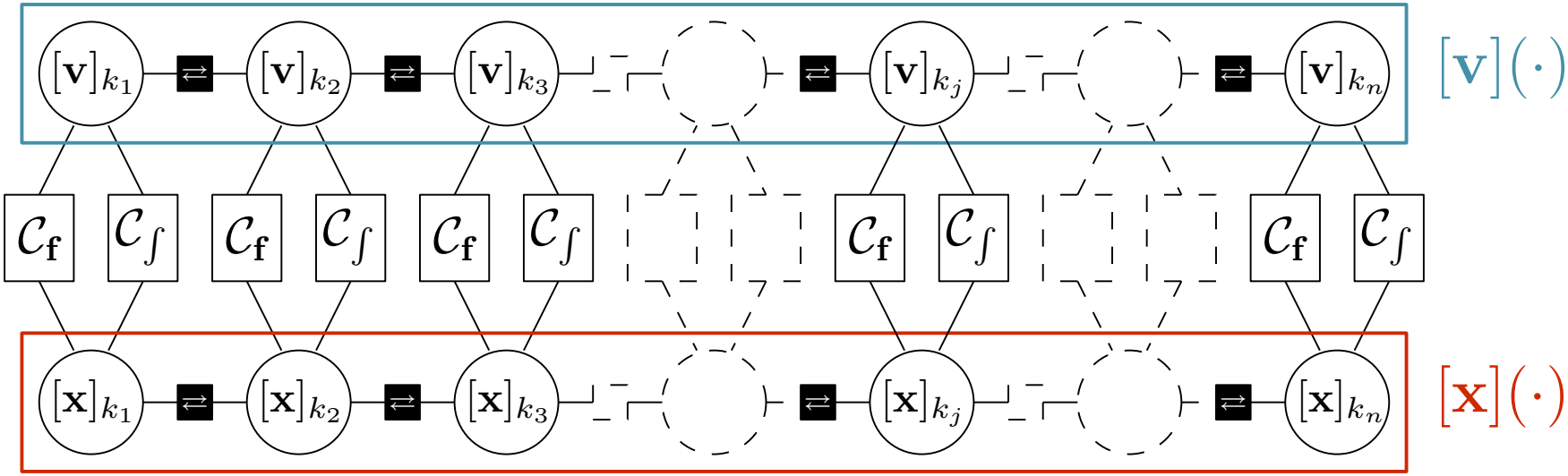

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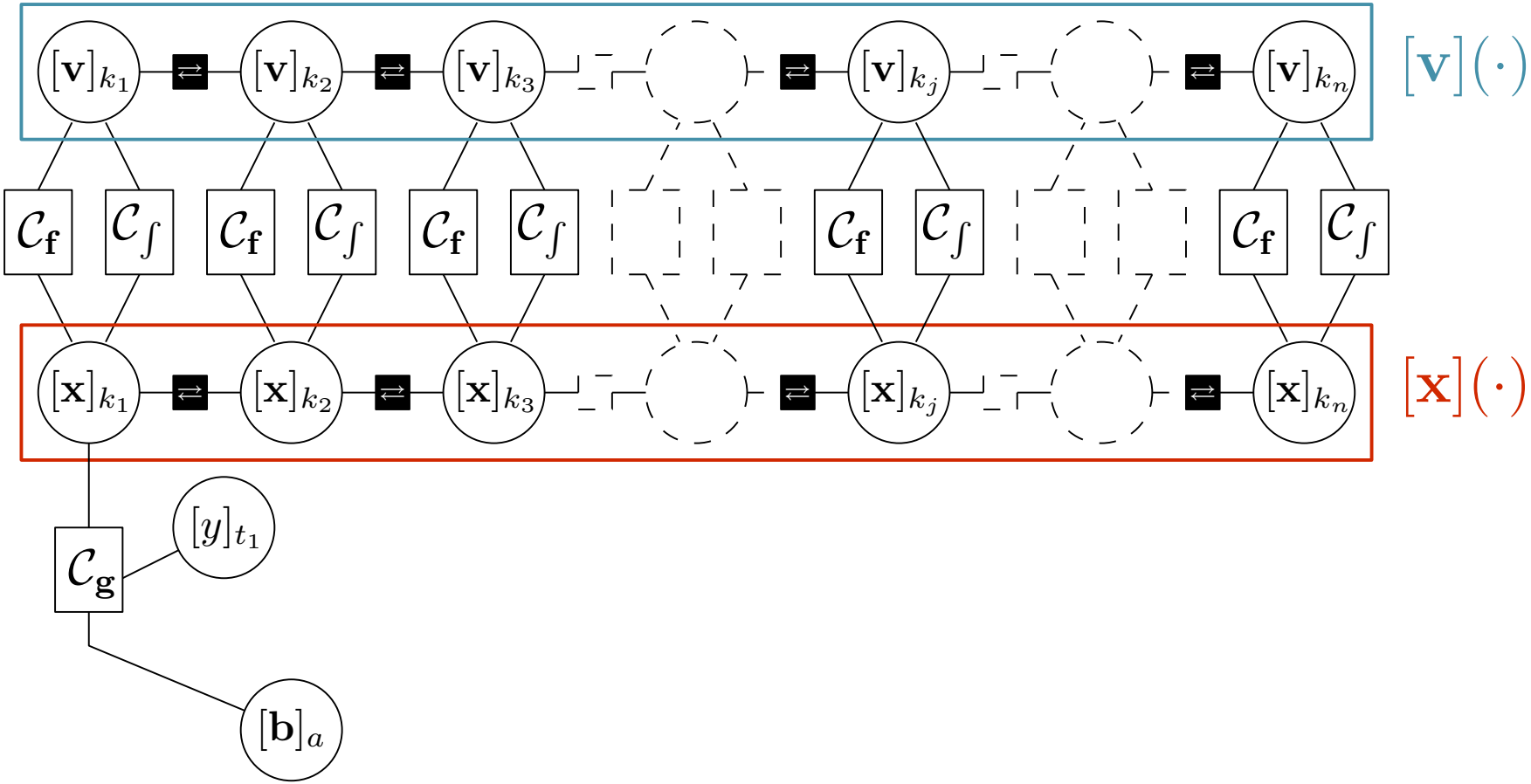
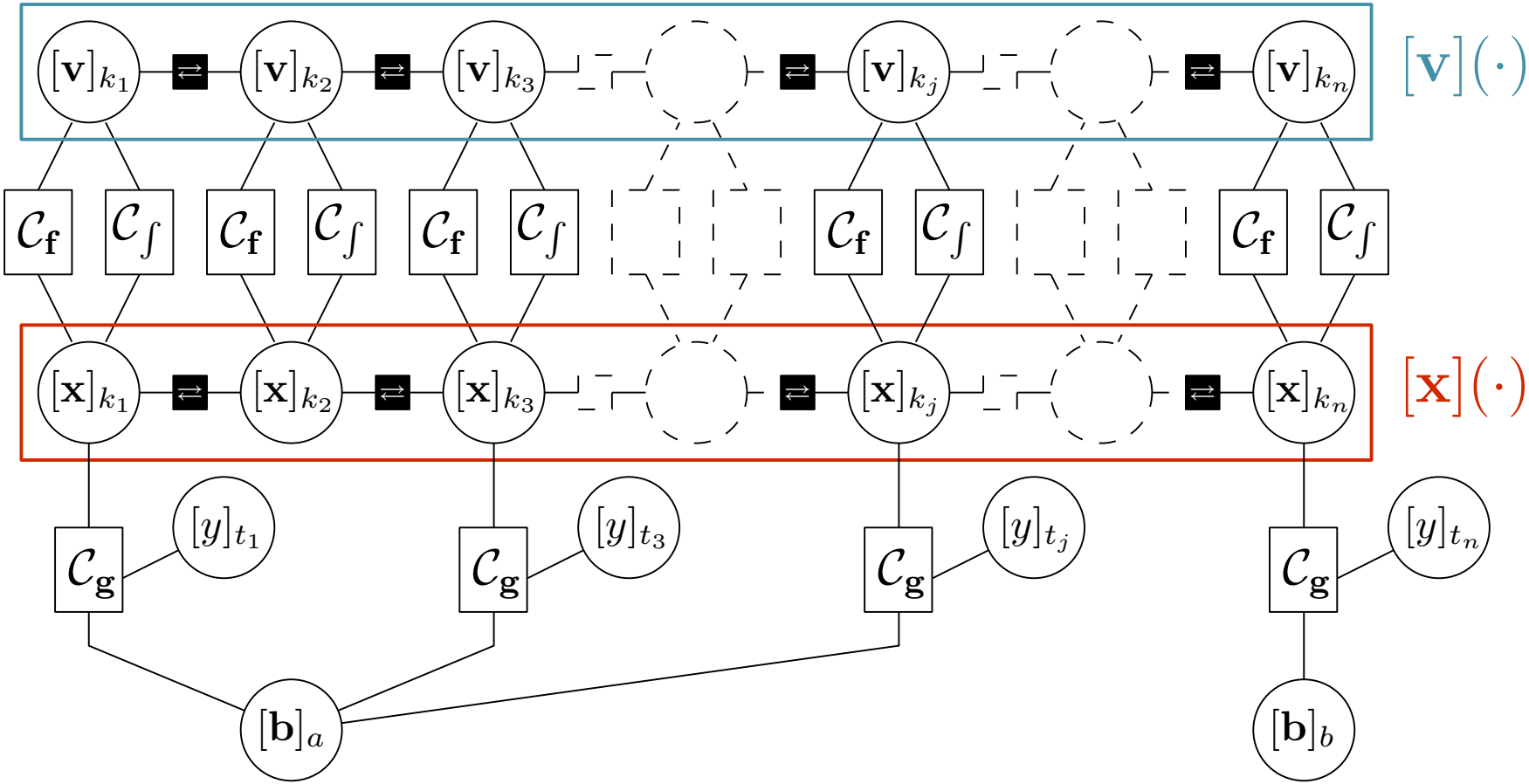


Graph of involved contractors and domains



Programming a SLAM-CN: methodology

1. Define domains:
   - intervals, boxes, tubes, ...
   - related to measurements or initialized as $[-\infty, \infty]$

..

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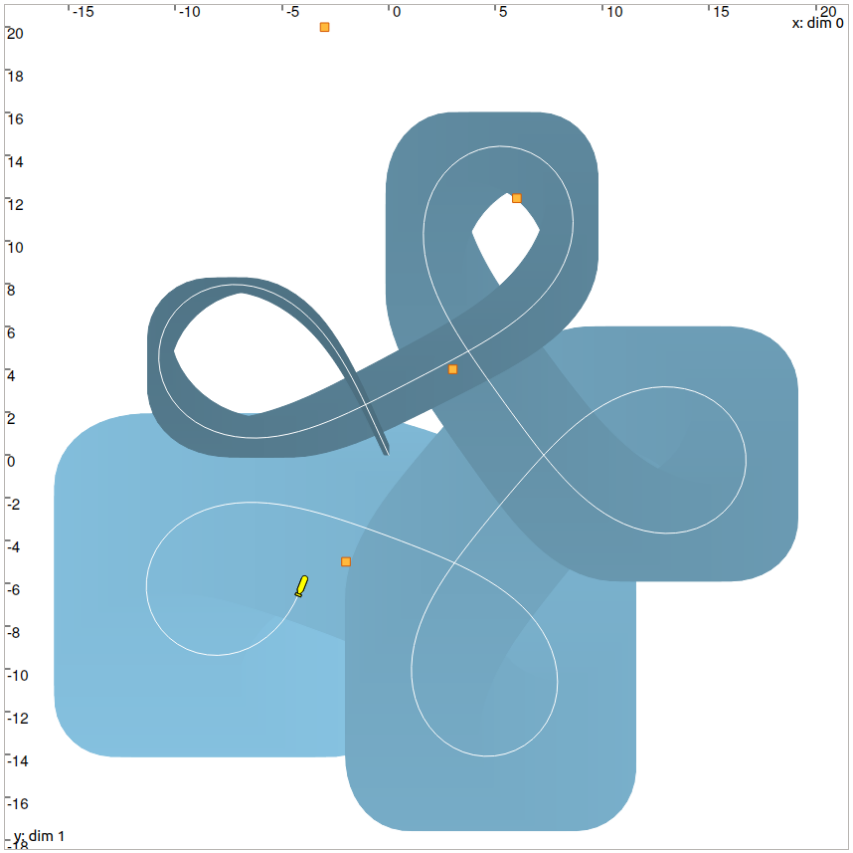
```
cn = ContractorNetwork()
cn.add(ctc_f, [x,v])
cn.add(ctc.deriv, [x,v])

for i in range(len(v_t)):
  pi = IntervalVector(4)
  cn.add(ctc.eval, [t[i],pi,x]
  cn.add(ctc.dist, [y[i],pi,b[i]])

cn.contract()
```

Programming a SLAM-CN: methodology




```
cn = ContractorNetwork()

cn.add(ctc.polar, ..
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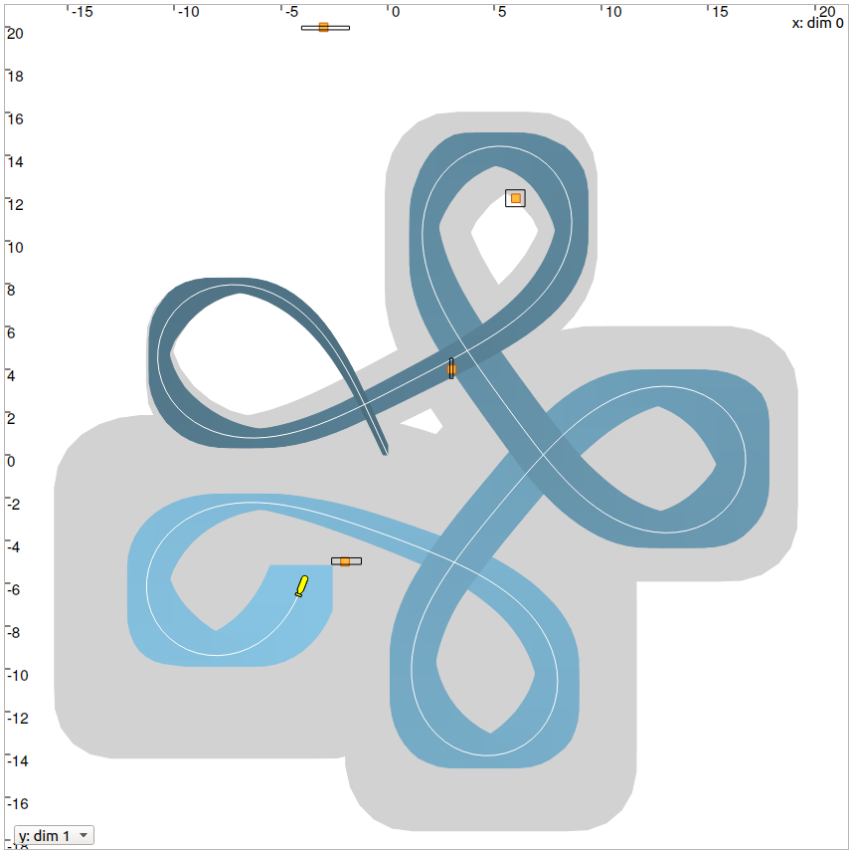


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```
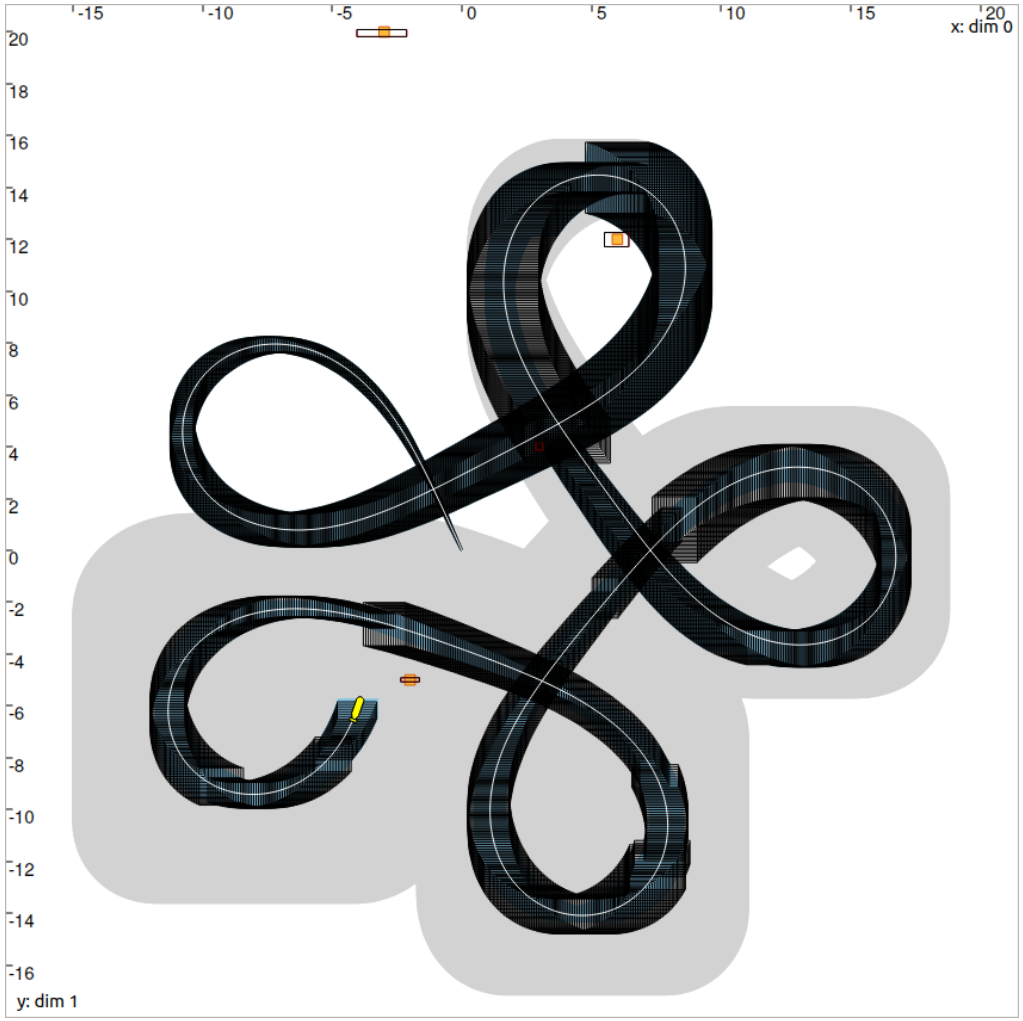

2399 contractors, 2410 dom.
Computation time: 0.25s

SLAM-CN: realtime application



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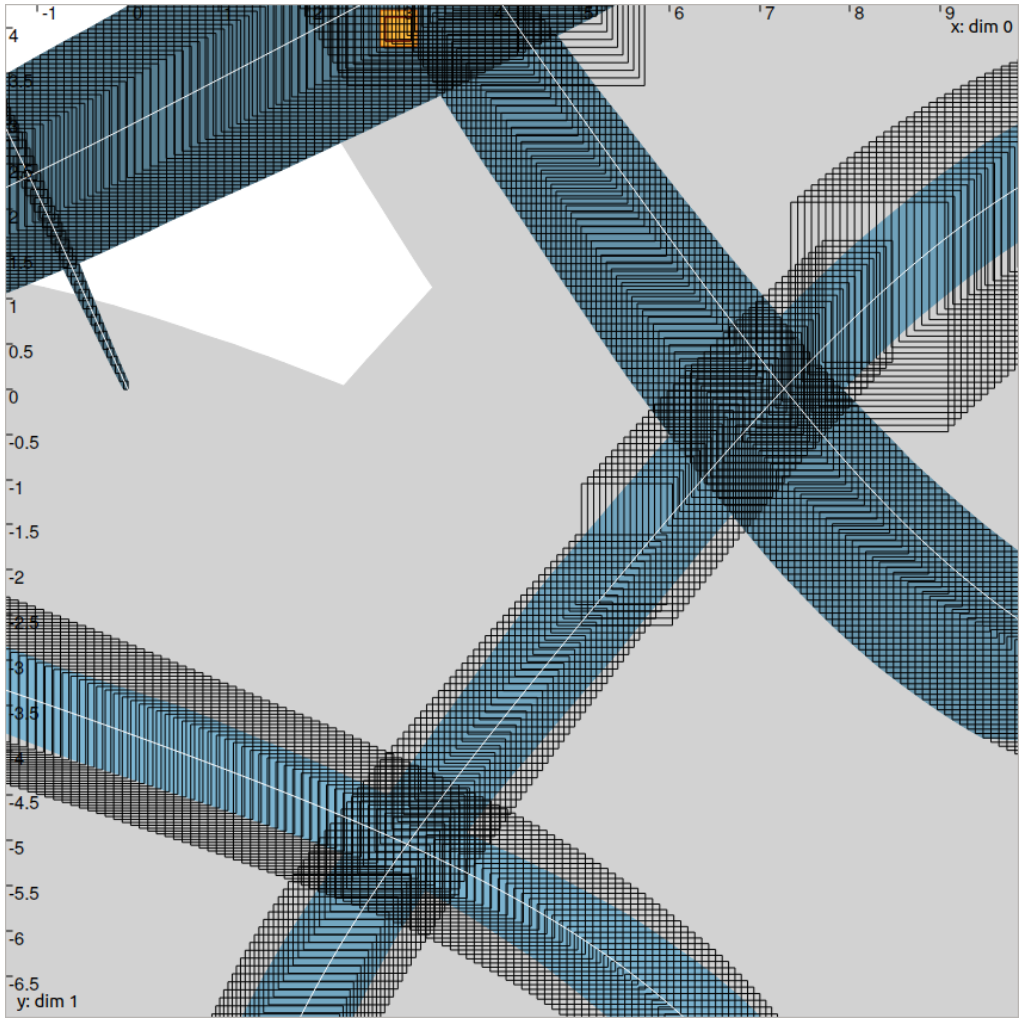

SLAM-CN: realtime application

Video